AUGUST

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

1 At the end of every month, Elle deposits \$500 into a savings account with an annual interest rate of 6 percent, compounded monthly. How much interest will be earned at the end of four years?

2 There are 100 members in the U.S. Senate (2 from each state). In how many ways can a committee of 5 senators be formed if no state may be represented more than once?

3 You have six sticks of lengths 10, 20, 30, 40, 50, and 60 cm. Find the number of noncongruent triangles that can be formed using three of these sticks as sides.

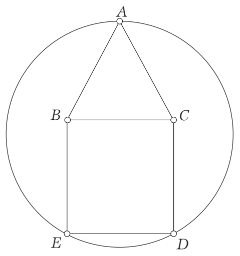
4 A glass box $7 \times 12 \times 18$ cm, closed on all six sides, is partially filled with colored water. When the box is placed on one of its 7×12 sides, the water level is 15 cm above the table. If the box is placed on one of its 7×18 sides, what will be the water level above the table, in centimeters?

5 Two integers are said to be partners if both are divisible by the same set of prime numbers. Find the number of positive integers less than 25 that have no partners less than 25.

6 There are four cottages on a straight road. The distance between Ted's and Alice's cottages is 3 km. Both Bob's and Carol's cottages are twice as far from Alice's as they are from Ted's. Find the distance between Bob's and Carol's cottages in kilometers.

7 Al, Bee, Cecil, and Di have \$16, \$24, \$32, and \$48, respectively. Their father proposed that Al and Bee share their wealth equally, and then Bee and Cecil do likewise, and then Cecil and Di. Their mother's plan is the same except that Di and Cecil begin by sharing equally, then Cecil and Bee, and then Bee and Al. Determine the number of children who end up with more money under their father's plan than under their mother's plan.

8 A pentagon made up of equilateral triangle ABC with side length 2 on top of square $BCDE$ is inside a circle passing through points A, D, and E. Find the radius of the circle.



9 When

$$\frac{19x - 8}{2x^2 - x - 21}$$

is decomposed into partial fractions (fractions that sum to the given fraction), what is the sum of the numerators when each fraction is reduced to lowest terms?

10 Two of the roots of the equation $2x^3 - 3x^2 + px + q = 0$ are 3 and -2. Find the third root of the equation.

11 How long is the side of the largest equilateral triangle that can be inscribed in a square with side length 1?

12 A round table can be made square by dropping the four leaves. If a side of the square table measures 36 inches, to the nearest hundredth of a square inch, how much smaller is the area of the table when the leaves are down than when the leaves are up?

13 A father and son eat meals together. If the son eats twice as fast as the father and the father eats a meal in 45 minutes, how many meals can the son eat in 3 hours?

14 Consider the circles that have radii $4\sqrt{5}$ and are tangent to the line $x - 2y = 20$ at the point $(6, -7)$. Find the sum of the x coordinates of the centers of the circles.

15 Given the equation $x^3 - 2x^2 + x - 3 = 0$, find an equation whose roots are each 2 less than the roots of the given equation.

16 An experiment consists of choosing with replacement an integer at random among the numbers from 1 to 9 inclusive. If we let M denote a number that is an integral multiple of 3 and N denote a number that is not an integral multiple of 3, arrange in order of increasing likelihood the following sequences of results: (a) $MNNMN$ (b) $NMMN$ (c) $NMMNM$ (d) $NNMN$ (e) $MNMM$

17 An 8-by-8-ft. area has been tiled with 1-foot-square tiles. Two of the tiles were defective. What is the probability that the two defective tiles share an edge?

18 Consider the equation $15x + 14y = 7$. Find the largest four-digit integer x for which there is an integer y so that the pair (x, y) is a solution.

19 Let P be the set of primes that divide 200!. What is the largest integer k so that the set of primes that divides k! is equal to P?

20 What is the remainder when $7^{348} + 25^{605}$ is divided by 8?

21 How many possible values can there be for three coins selected from among pennies, nickels, dimes, and quarters?

22 In a trapezoid $ABCD$ with $\overline{AB}$ parallel to $\overline{CD}$, the diagonals intersect at point E. The area of triangle ABE is 32, and the area of triangle CDE is 50. Find the area of the trapezoid.

23 Find the number of four-digit positive integers divisible by either 3 or 7.

24 What is the smallest positive integer that when divided by 10, 9, 8, 7, and 6 leaves the remainders 9, 8, 7, 6, and 5, respectively?

25 If the product of three numbers in geometric progression is 216 and their sum is 19, find the largest of the three numbers.

26 Among all collections, S, of positive integers whose sum is 28, what is the largest product that the integers in S can form?

27 Consider the set T of positive integers d for which there exists an integer n such that d evenly divides both $(13n + 6)$ and $(12n + 5)$. Find the sum of the elements of T.

28 Suppose that x and y are two real numbers such that $x - y = 2$ and $x^2 + y^2 = 8$. Find $x^3 - y^3$.

29 What is the remainder when $1! + 2! + 3! + 4! + \cdots + 99! + 100!$ is divided by 18?

30 Find n so that

$$\frac{1}{1+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}} = 100.$$

31 If $1 \le a < b < c < d < e \le 9$, what arrangement of whole numbers a, b, c, d, and e into a three-digit and a two-digit number will produce the greatest product? Use each digit only once.

SOLUTIONS to calendar

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Problems in this month's calendar are from the North Carolina State Mathematics Contest, 2004.

The Editorial Panel of the *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly "Calendar." Send problems to the "Calendar" editors. Remember to include a complete solution for each problem submitted.

Another source of problems in calendar form available from NCTM is *Calendar Problems from the* Mathematics Teacher (a book featuring more than 400 problems, organized by topic, order number 12509, $22.95). Individual members receive a 20 percent discount off this price. A catalog of educational materials is available at www.nctm.org.—Eds.

1. \$3048.92. The interest on \$500 after k months is $500 \cdot (1.005^k - 1)$. The total interest is $500 \cdot \{1.005^{47} - 1 + 1.005^{46} - 1 + \cdots + 1.005^0 - 1\} =$

$$500\left(\frac{1.005^{48}-1}{1.005-1}-48\right)=3048.92.$$

2. 67,800,320. Choose five of fifty states,

$$\binom{50}{5} \text{ ways,}$$

and one of two senators from each of the five states, 2^5 ways,

$$\binom{50}{5}2^5 = 67{,}800{,}320.$$

Alternatively, the first senator may be chosen in 100 ways, the second in 98, the third in 96 and so on. Once five senators have been chosen, divide by 5! because the order of choice is unimportant.

$$\frac{100 \bullet 98 \bullet 96 \bullet 94 \bullet 92}{5!} = 67{,}800{,}320$$

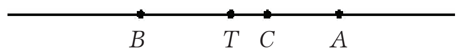
3. 7. By the triangle inequality, the possible lengths of sticks that form triangles are (20, 30, 40), (20, 40, 50), (20, 50, 60), (30, 40, 50), (30, 40, 60), (30, 50, 60), and (40, 50, 60).

4. 10 cm. If the water is x cm above the table when the box is placed on the 7×18 side, then the volume of water = $7 \times 12 \times 15 = 7 \times 18 \times x$, so $x = 10$ cm.

Alternate solution. As described, the water level is 15/18 or 5/6 of the height; thus, when you turn the box so that the height is 12, the water level will be 5/6 of 12 = 10 cm.

5. 12. Integers without partners are 1, 5, 7, 11, 13, 14, 15, 17, 19, 21, 22, 23.

6. 4. As shown in the figure, without loss of generality, we may assume Carol lives between Alice and Ted, 1 km from Ted and 2 km from Alice. Bob's cottage is 3 km from Ted's, on the other side of Alice's cottage and 6 km from Alice. Since Bob's cottage is 3 km from Ted's and Carol's is 1 km from Ted's, the distance between Bob's and Carol's cottages is 4 km.



7. 2. Under their father's plan, Al will have

$$\frac{16+24}{2} = \$20,$$

Bee will have

$$\frac{20+32}{2} = \$26,$$

Cecil will have

$$\frac{26+48}{2} = \$37,$$

and Di will also have \$37.

Under their mother's plan, Di will have

$$\frac{48+32}{2} = \$40,$$

Cecil will have

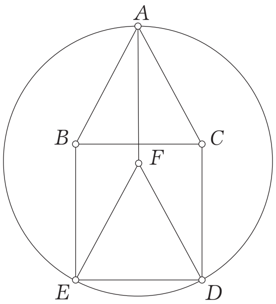
$$\frac{40+24}{2} = \$32,$$

Bee and Al will both have

$$\frac{32+16}{2} = \$24.$$

So Bee and Cecil will end up with more money under their father's plan.

8. 2. Consider the equilateral triangle EDF with F inside the square. Since the measure of angle ABE is 150 degrees, it follows that $ABEF$ is a parallelogram with each side of equal length (i.e., a rhombus). Since F is equidistant from A, D, and E, it is the center of the circle, and the radius is 2.



9. 14.

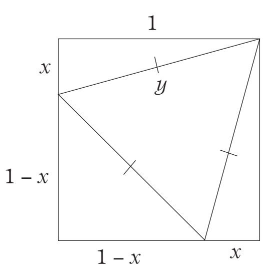
$$\frac{A}{2x-7}+\frac{B}{x+3}=\frac{A(x+3)+B(2x-7)}{(2x-7)(x+3)}$$
$$=\frac{19x-8}{2x^2-x-21}$$

leads to $A = 9$ and $B = 5$.

10. 1/2. The sum of the roots is the opposite of the coefficient of x^2 of the associated polynomial with lead coefficient of 1, i.e., 3/2. Thus, the third root is 1/2.

Alternate solution. Substituting the given values for the roots into the equation produces a system of linear equations that can be solved for $(p, q) = (-11, 6)$. Then the cubic equation can be solved in any number of ways.

11. $2\sqrt{2-\sqrt{3}}$. From the right triangle in the upper left corner (see diagram): $1^2 + x^2 = y^2$. From the right triangle in the lower left corner: $(1 - x)^2 + (1 - x)^2 = y^2$. Thus, $x = 2 - \sqrt{3}$ (reject $2 + \sqrt{3}$ because $1 - (2 + \sqrt{3}) < 0$), and $y = \sqrt{1 + (2 - \sqrt{3})^2} = 2\sqrt{2-\sqrt{3}}$.



12. 739.75. The diameter of the circle = diagonal of the square = $36\sqrt{2}$. So the difference in areas is

$$\pi\left(\frac{36\sqrt{2}}{2}\right)^2-36^2\approx 739.75.$$

13. 8. Since the father can eat 4 meals in 3 hours, and the son eats twice as fast, the son can eat 8 meals in the same time.

14. 12. The centers will be symmetrically placed along a line through $(6, -7)$ perpendicular to $x - 2y = 20$. Thus the average values of the x coordinates will be 6.

15. $x^3 + 4x^2 + 5x - 1$. To obtain the graph of the new equation, shift the graph of the original equation left by two units. Algebraically, this corresponds to replacing x with $x + 2$ to obtain $(x + 2)^3 - 2(x + 2)^2 + x + 2 - 3 = x^3 + 4x^2 + 5x - 1$.

16. c, e, a, b, d. Since there are 3 multiples of 3 among the digits 1 to 9, the probability that a digit is of type M is 1/3, while the probability that it is of type N is 2/3. The probability of each of the sequences occurring is

$$Pr(MNNMN)=\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3=\frac{8}{243},$$

$$Pr(NMMN)=\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2=\frac{4}{81}=\frac{12}{243},$$

$$Pr(NMMNM)=\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2=\frac{4}{243},$$

$$Pr(NNMN)=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3=\frac{8}{81}=\frac{24}{243},$$

$$Pr(MNMM)=\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)=\frac{2}{81}=\frac{6}{243}.$$

17. 1/18. There are

$$\binom{8^2}{2}=\frac{8^2(8^2-1)}{2}\text{ ways}$$

of choosing two arbitrary squares for the two defective tiles. If the two defective tiles share an edge, then two cases can be considered.

Case 1. One of the tiles was placed in any of the top 7 rows (8×7 ways), and the other was placed in the square below.

Case 2. One tile was placed in any of the left 7 columns (8×7 ways), and the other was placed in the square to its right. So the probability is

$$\frac{2\times 8\times 7}{\left(\frac{64\times 63}{2}\right)}=\frac{1}{18}.$$

Alternate solution. The first (defective) tile may be placed within the interior, edge, or corner with probabilities of 36/64, 24/64, and 4/64, respectively. The second tile would have the following respective probabilities of being adjacent: 4/63, 3/63, and 2/63. Thus, the probability is the sum of the products of the respective probabilities of each case, which is 224/4032, or 1/18.

Second alternate solution. Look at a simpler problem. In a 2-by-2 ft. area, there are 4 ways to place the tiles successfully. If the area is 3 by 3, there are 12 successful ways to place the tiles; and 4-by-4 and 5-by-5 areas respectively yield 24 and 40 successful ways. From second differences, the relation is seen to be quadratic, $y = n^2 - 2n$, where n represents the length of one side of the tiled area. Evaluate at $n = 8$ and divide by ${}_{64}C_2$.

18. 9989. $15(7) + 14(-7) = 7$, so $(7, -7)$ is a solution. For $x = 7 + s$ to be a solution,

$$y=\frac{7-15x}{14}$$
$$=\frac{7-15(7+s)}{14}$$
$$=-7-\frac{15s}{14}$$

must be an integer, so 14 must divide $15s$. Since 14 and 15 are relatively prime, s must be a multiple of 14. Thus, $s = 14t$ for some integer t and $x = 7 + 14t$. If $x = 7 + 14t \le 9999$, then the largest $t = 713$ and $x = 9989$.

Alternate solution. Since $15x + 14y = 7$ can be rewritten as $x + 14x + 14y = 7$, associating the addition and dividing tells us that 7 must divide x. Further,

$x/7 + 2(x + y) = 1$ tells us that the quotient of x and 7 must be odd (odd + even = 1, an odd). The largest four-digit number divisible by 7 is 9996 but even; next largest is 9989.

19. 210. $P = \{p \text{ prime} \mid p \le 200\} = \{2, 3, \ldots, 197\}$. Since 211 is the smallest prime greater than 200, these are the only primes that will divide all integers $\le k!$ for $k = 210$.

20. 2. $7^2 \equiv 1 \pmod 8$, so $7^{348} = (7^2)^{174} \equiv 1^{174} \pmod 8 \equiv 1 \pmod 8$. $25 \equiv 1 \pmod 8$ so $25^{605} \equiv 1^{605} \pmod 8 \equiv 1 \pmod 8$. Thus $7^{348} + 25^{605} \equiv 1 \pmod 8 + 1 \pmod 8 \equiv 2 \pmod 8$. The remainder is 2.

21. 20.

Case 1. All three are of the same denomination: 4 ways
Case 2. Two coins are of any one of the four denominations, the third is of any of the remaining three denominations: $4 \times 3 = 12$ ways
Case 3. All three are of different denominations:

$$\binom{4}{3} = 4 \text{ ways}$$

Adding up the number of ways for all three cases, we get $4 + 12 + 4 = 20$ ways. One checks that the values given by these 20 ways are all different.

22. 162. Since $\overline{AB}$ is parallel to $\overline{CD}$, $\triangle EAB$ is similar to $\triangle ECD$. Let FEG be the line through E perpendicular to $\overline{AB}$ and $\overline{CD}$, intersecting $\overline{AB}$ at F and $\overline{CD}$ at G. Let

$$\frac{AB}{CD} = \frac{FE}{GE} = \sqrt{\frac{32}{50}} = \frac{4}{5}$$

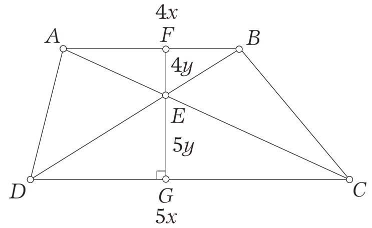
(corresponding sides of similar figures are in the same ratio as the square roots of their areas). Let $AB = 4x$, $CD = 5x$, $FE = 4y$, $EG = 5y$. Then the area of the trapezoid

$$= \frac{1}{2}(FE + EG)(AB + CD)$$
$$= \frac{1}{2}(4y + 5y)(4x + 5x)$$
$$= \frac{81}{2}xy.$$

But the area of triangle ABE is

$$32 = \frac{1}{2}(FE)(AB)$$
$$= \frac{1}{2}(4y)(4x).$$

Thus, $xy = 4$, and the area of the trapezoid is 162.



23. 3857. Let $[x]$ be the largest integer less than or equal to x. Then there are

$$\left[\frac{9999}{n}\right]$$

positive integers ≤ 9999 divisible by n, of which

$$\left[\frac{999}{n}\right]$$

are ≤ 999, so there are

$$\left[\frac{9999}{n}\right] - \left[\frac{999}{n}\right]$$

four-digit positive integers divisible by n. Letting $n = 3$, there are

$$\left[\frac{9999}{3}\right] - \left[\frac{999}{3}\right] = 3000$$

four-digit positive integers divisible by 3. Similarly, there are 1286 divisible by 7. This includes

$$\left[\frac{9999}{21}\right] - \left[\frac{999}{21}\right] = 429$$

that are divisible by both 3 and 7 and so are counted twice. So there are $3000 + 1286 - 429 = 3857$ four-digit positive integers divisible by 3 and/or 7.

Alternate solution. Find the number of terms in the arithmetic sequences from 1002 to 9999 with common difference 3, from 1001 to 9996 with difference 7, and from 1008 to 9996 with difference 21, adding the first two and subtracting the third.

[*This month's "Delving Deeper" contains a discussion of divisibility tests for 3 and 7.—Ed.*]

24. 2519. Let n be the integer to be found. Since $n = 10x + 9 = 10(x + 1) - 1$, we note that $n + 1$ is a multiple of $10 = 2(5)$. Similarly, $n + 1$ is a multiple of $9 = 3(3)$, $8 = 2^3$, 7, $6 = 2(3)$. So $n + 1$ must contain the factors 2^3, 3^2, 5, and 7. Thus, $n + 1 = 2^3 3^2(5)(7) = 2520$, and $n = 2519$.

25. 9. Let the 3 numbers be x/r, x, rx. Then $x^3 = 216$ and $x = 6$.

$$\frac{6}{r} + 6 + 6r = 19$$

leads to $x = 2/3$ or $3/2$ and the sequences 4, 6, 9 or 9, 6, 4.

26. 26,244. If $n \ge 4$ is in S, then $2(n - 2) \ge n$. Thus, the largest product with a given sum occurs with no integers larger than 3. Since $2^3 < 3^2$, two 3s give a larger product than three 2s. Since $3 \times 1 < 2^2$, we should choose two 2s instead of a 3 and a 1. Thus, $28 = 3(8) + 2(2)$, and the largest product is $3^8 2^2 = 26{,}244$.

27. 8. Since d divides both $13n + 6$ and $12n + 5$, d divides $12(13n + 6) - 13(12n + 5) = 7$, so d can only be 1 or 7. If $n = 6$, $12n + 5 = 77$ and $13n + 6 = 84$ are both divisible by 1 and 7. Thus, $T = \{1, 7\}$.

28. 20. $2^2 = (x - y)^2 = x^2 - 2xy + y^2 = 8 - 2xy$, so $xy = 2$. Thus, $x^3 - y^3 = (x - y)(x^2 + xy + y^2) = 2(8 + 2) = 20$.

29. 9. $18 = 3 \times 6$, so 18 evenly divides $n!$ for $n \ge 6$. Thus, the remainder when $1! + 2! + 3! + 4! + \cdots + 99! + 100!$ is divided by 18 is the same as the remainder when $1! + 2! + 3! + 4! + 5! = 1 + 2 + 6 + 24 + 120 = 153 = 18(8) + 9$ is divided by 18. So the remainder is 9.

30. 20,200. After rationalizing denominators the left side becomes

$$\frac{1-\sqrt{3}}{1-3} + \frac{\sqrt{3}-\sqrt{5}}{3-5} + \frac{\sqrt{5}-\sqrt{7}}{5-7} + \cdots + \frac{\sqrt{2n-1}-\sqrt{2n+1}}{(2n-1)-(2n+1)}$$
$$= \frac{1-\sqrt{2n+1}}{-2}.$$

Solving

$$\frac{1-\sqrt{2n+1}}{-2} = 100,$$

we get $2n + 1 = (200 + 1)^2$. So $n = 20{,}200$.

31. $\begin{array}{r} dca \\ \underline{\times eb} \end{array}$ (875×96)

Consider the general form of the problem:

$$\begin{array}{r} p_1p_2p_3 \\ \underline{\times m_1m_2} \end{array}$$

where the value of $p_1p_2p_3 = 100p_1 + 10p_2 + 1p_3$ and the value of $m_1m_2 = 10m_1 + 1m_2$. The two largest values, e and d, should occur as lead digits to ensure the largest product, and we assign $e = m_1$ (as opposed to p_1), since it will distribute over the remaining digits of $p_1p_2p_3$—that is, $10e \cdot (100d + 10p_2 + 1p_3) > 10d \cdot (100e + 10p_2 + 1p_3)$. By similar logic, assign the least value, a, to p_3. This leaves only b and c to assign to positions p_2 and m_2. Assume the differences between a, b, c, d, and e to be n_1, n_2, n_3, and n_4, respectively (i.e., $a + n_1 = b$, etc.). It remains to show that the product

$$\begin{array}{r} dca \\ \underline{\times eb} \end{array}$$

is greater than

$$\begin{array}{r} dba \\ \underline{\times ec} \end{array}$$

This amounts to showing that $100ec + 100bd + 10cd + ab - (100eb + 100cd + 10cb + ac) > 0$.

$$100ec - 100eb = 100e\,n_2$$
$$100bd - 100cd = 100d\,(-n_2)$$
$$100e\,n_2 - 100d\,(n_2) = 100\,n_4\,n_2 > 0$$
$$10cd - 10cb = 10c\,(n_2 + n_3) > 0$$
$$ab - ac = a\,(-n_2)$$

However, the magnitude of this negative product cannot surpass the other addends. ∞

Write for a Department!

Which department do you always read first? "Calendar?" "Media Clips"? "Technology Tips"? How many times have you thought—

- "I have a great problem for the "Calendar,"
- "My file is bulging with newspaper clippings for bringing real-world mathematics into the classroom," or
- "Just yesterday, I thought of a new calculator approach."

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