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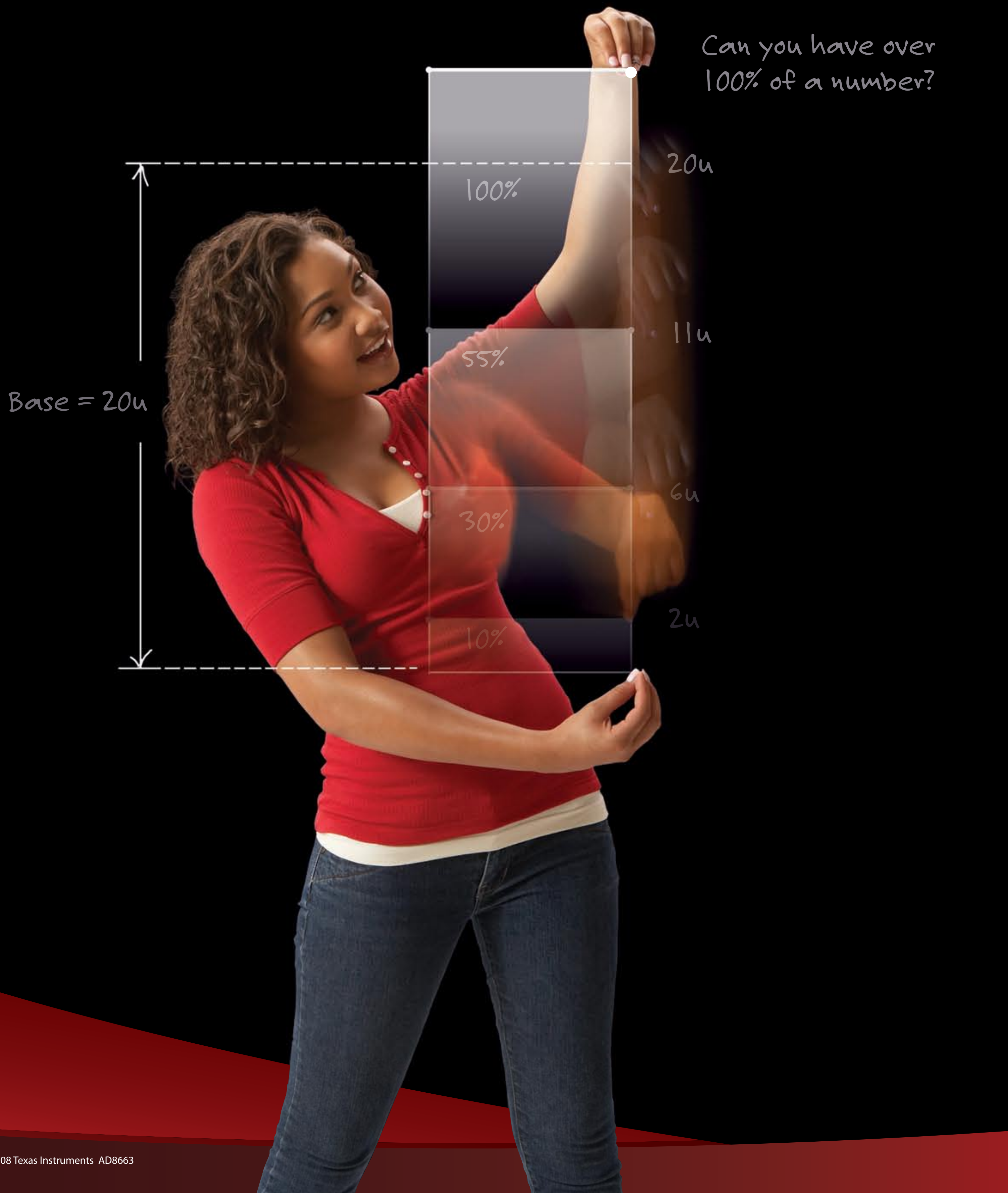

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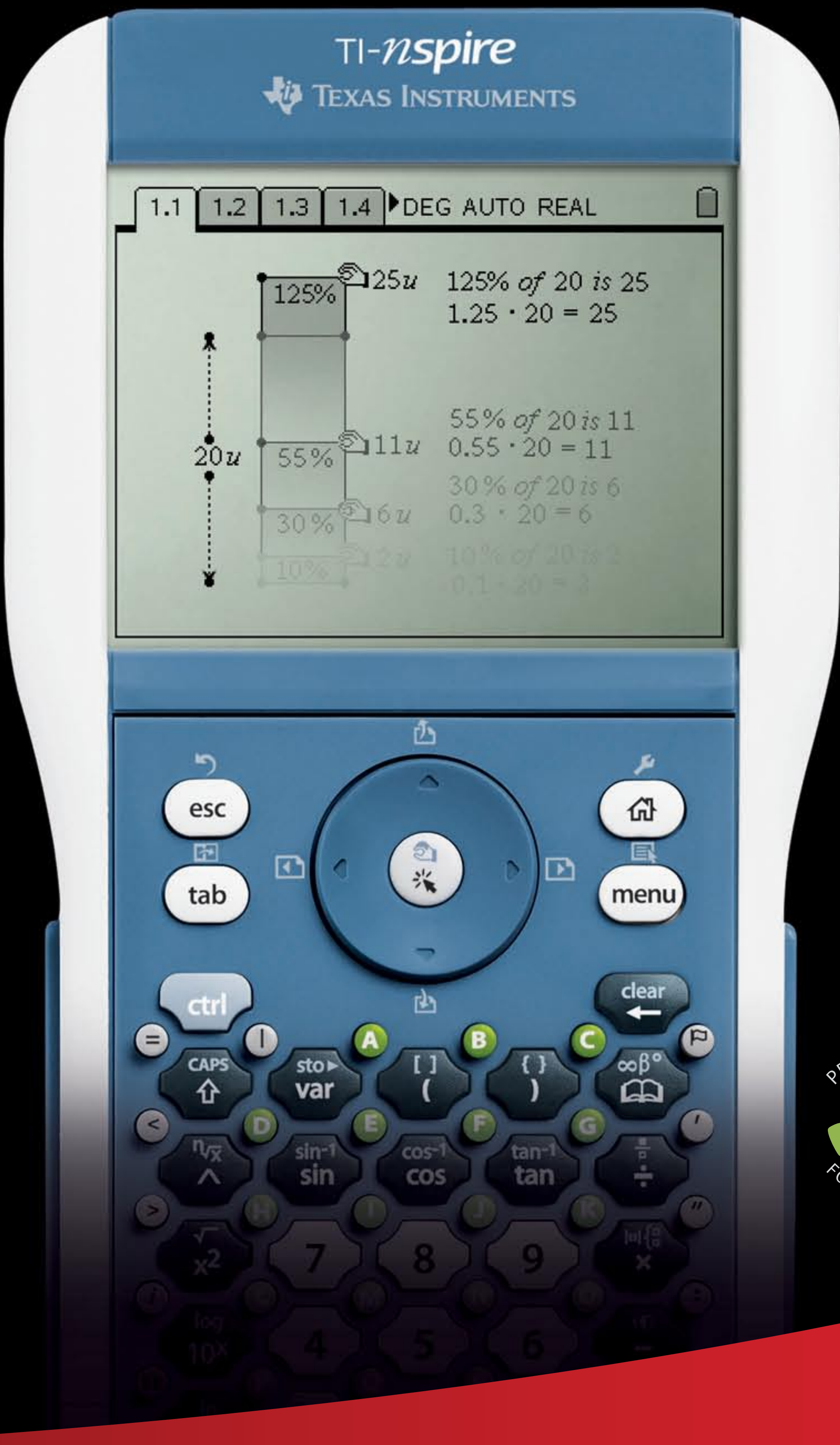

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Probability in Practice

The Case of Turkey Bingo

Christopher Danielson and Eric Jenson

The Wednesday before Thanksgiving presents a challenge to teachers in many U.S schools. Some students are absent because they are traveling to be with their extended families for the holiday. Other students, assuming that nothing important will happen at school when so many of their peers are absent, may also be absent. One school's solution to the attendance problem on this day is to have a daylong bingo game. This article recounts a mathematics problem that arose in this schoolwide game in one recent year, explores the mathematics behind solving the problem, and briefly examines two technological approaches to the solution.

This problem provides an example of a type of probability analysis we recommend doing with high school students. In particular, we show how to use a software package—Fathom—to simulate a probability situation that is too complicated for direct theoretical analysis, and we compare the results with an estimated computation. The techniques described here could be used to solve a variety of interesting probability problems.

THE GAME

During the first period of the school day, students construct a Turkey Bingo card by completing a 5 × 5 grid with selections chosen from a common list of 40 seasonal words (e.g., *turkey*, etc.). During the year discussed, 5 words were to be read out at the end of the first, second, third, and fifth periods, making a total of 20 words in play.

Turkey Bingo differs from the traditional bingo game in three important ways (see **fig. 1**):

(a)

turkey	Thanks-giving	cran-berries	stuffing	Thursday
Pilgrims	Indians	thanks	harvest	fall
travel	family	friends	weekend	pumpkins
apples	yams	mashed potatoes	green beans	pie
football	indoors	white linen tablecloths	school vacation	Lincoln

(b)

Fig. 1 A typical bingo card (**a**) and a typical Turkey Bingo card (**b**)

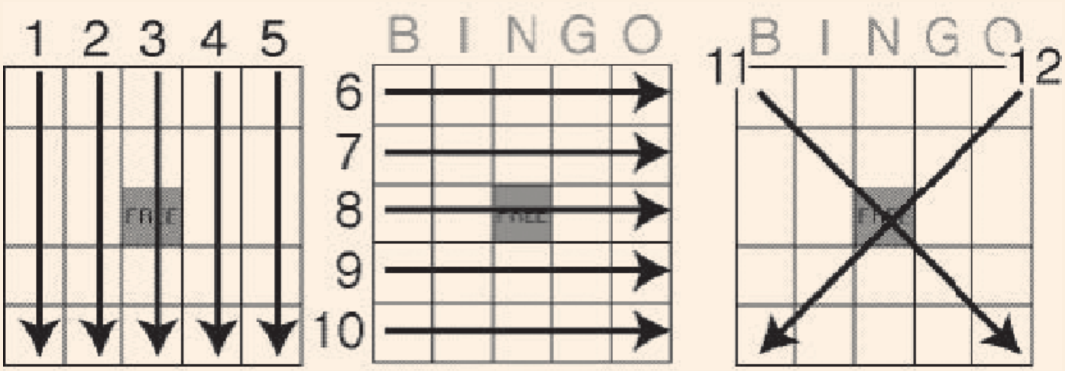

Fig. 2 On a standard bingo card, there are 12 winning lines, and lines 3, 8, 11, and 12 each include the free space. In Turkey Bingo, there is no free space, but a player wins in Turkey Bingo in the same way as in traditional bingo.

1. Turkey Bingo offers no free space. Each student must choose 25 words to fill in the squares on the grid.
2. The numbers on a standard bingo card are restricted to specific columns: Numbers 1 through 15 must go in the B column, numbers 16 through 30 must go in the I column, and so on. In Turkey Bingo, there are no restrictions on where the words are placed, except that exactly one word goes in each square and a word may not be repeated on a card.
3. Students fill out their own Turkey Bingo cards, so we cannot be sure that all cards are different from one another, nor can we be sure that two or more cards do not share important characteristics (such as having the word *turkey* in the center square). In standard bingo, players are given preprinted cards to ensure variation.

Words are chosen randomly and read over the school's public address system. Each student colors in the corresponding square on his or her grid, as in traditional bingo. The object is to get five shaded squares in a horizontal row, in a vertical column, or on one of the two main diagonals (see **fig. 2**). (The cover-all variation, in which all spaces of the card must be covered to win, and other variations are not part of Turkey Bingo.) The game continues through the end of the day rather than finishing with the first winner. Winners receive five dollars to spend at the school store.

THE PROBLEM

Co-author Eric Jenson was in his first year of teaching mathematics at this school when he encountered Turkey Bingo. Following the first period of the Wednesday before Thanksgiving that year, 5 words were randomly drawn and read over the public address system as planned. However, before the next 5 words were read after the second period, a change in the rules of the game was announced: Because many students already had 4 words in a row after just the first period, only one word would be drawn and read after the subsequent periods.

Jenson was certain that 5 words per period would be too many—resulting in many, many winners—but that switching to 1 word per period (for a total of 8 words: the 5 from first period, the 1 from second period, and 1 for each of the remaining two periods) would not be enough to yield a substantial number of winners. He called the office and learned that the game's sponsors were prepared for as many as 20 winners in this school of approximately 1000 students. Quickly, Jenson had his class of eleventh graders estimate probabilities of winning Turkey Bingo on the basis of the number of words drawn. The class determined that if 12 words were to be drawn, there would be 14 winning lines on 1000 Turkey Bingo cards and recommended that 3 words be read after each of the two remaining periods. Before elaborating on the class's estimation techniques and providing the school's results, we examine the mathematics of bingo and Turkey Bingo.

BINGO MATH

Bingo as a mathematical activity has been the subject of several *Mathematics Teacher* articles over the years. Bay et al. (2000) analyzed the probabilities of winning several variations of bingo on the fewest number of draws (e.g., five draws for a vertical row); Mercer (1993) analyzed the probabilities of winning cover-all bingo on various numbers of draws; and Catlett (1991) outlined a 3×3 bingo game to be used for practicing and reviewing computational skills. This article introduces a variation on bingo and considers the application of popular computer technology that did not exist when the previous articles were written or was not the articles' focus.

Traditional Bingo

Although counting possible cards in traditional bingo is straightforward, the probabilities associated with winning are complex. The B column has five empty squares and 15 numbers to choose from (without replacement), and the numbers are randomly ordered from top to bottom. This yields $15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 = 360{,}360$ possible first columns. The setup is the same in columns 2 (I), 4 (G), and 5 (O), but column 3 (N) has a free space, leaving only four squares to be filled in. This yields $15 \cdot 14 \cdot 13 \cdot 12 = 32{,}760$ possible third columns. In total, there are $(360{,}360)^4 \cdot 32{,}760 \approx 5.52 \cdot 10^{26}$ possible bingo cards.

Each card has 12 possible winning lines (see **fig. 2**). The vertical winning lines (indicated as numbers 1–5 in **fig. 2**) are counted above, except that we do not care about order within the column—for instance, {1, 2, 3, 4, 5} will win whenever {2, 3, 4, 5, 1} does. Therefore, there are only

$$\binom{15}{5} = 3{,}003$$

different winning combinations in the B, I, G, and O columns, and only

$$\binom{15}{4} = 1{,}365$$

winning combinations in the N column. Possible horizontal lines (numbers 6–10 in **fig. 2**) are more numerous because the columns are independent of one another. Consider the top row: There are 15 possibilities for the first square (in the B column), 15 for the second square, and so on. Thus, there are $15^5 = 759{,}375$ possible first rows. Although the squares within each row are independent of one another, the rows themselves are not independent; once we have chosen the first row, we have fewer choices for each square in the second row. In fact, there are $14^5 = 537{,}824$ second rows, $13^4 = 28{,}561$ third rows (recall the free space), $12^4 \cdot 13 = 269{,}568$ fourth rows, and $11^4 \cdot 12 = 175{,}692$ fifth rows. Note that this analysis leads to the same total number of bingo cards as counting columns yielded above:

$$\begin{aligned} &15^5 \cdot 14^5 \cdot 13^4 \cdot (12^4 \cdot 13) \cdot (11^4 \cdot 12) \\ &\quad = (15 \cdot 14 \cdot 13 \cdot 12 \cdot 11)^4 \cdot (15 \cdot 14 \cdot 13 \cdot 12) \\ &\quad \approx 5.52 \cdot 10^{26} \end{aligned}$$

The possibilities for each main diagonal, because the squares are independent of one another and include the center free space, are the same as for the third row.

To complicate matters, the rows, columns, and diagonals of a traditional bingo card are not independent. Once we have chosen the columns, the rows and the diagonals are fixed. Each square is part of at least two possible wins: a row and a column. Eight of the 24 nonfree squares are part of three possible wins: a row, a column, and a diagonal.

There are many questions we can ask about traditional bingo, including these:

- How many winning lines can we expect with x draws?
- How many winning cards can we expect with x draws?
- What is the probability of card y winning in x draws?

Each of these questions requires a different sort of analysis. Although these analyses are not the focus of this article, the technology-based estimation techniques described here could easily be adapted to answer them.

Turkey Bingo

Turkey Bingo introduces the simplification that the squares are completely independent of one another, with the exception of nonreplacement. No particular words are restricted to particular columns. As a result, there are

$$\binom{40}{25} = 40{,}225{,}345{,}056$$

ways to choose the words and $25! \approx 1.55 \cdot 10^{25}$ ways to organize them, yielding approximately $6.24 \cdot 10^{35}$ possible cards. As with traditional bingo, however, the resulting columns, rows, and diagonals are not independent of one another. In the following analyses, we will assume that students fill out their Turkey Bingo cards randomly and that any two cards are independent of each other. Biased behavior on the part of students (such as a tendency to put the word *turkey* in the center square) would affect the calculations we perform here.

The question of particular interest in Jenson's school was this: How many words should be drawn so that the expected number of winners is close to 20, without going over? The next section outlines two uses of technology to answer this question.

TWO APPROACHES TO THE PROBLEM

Currently, the relationship between experimental probability and theoretical probability is a standard topic in the secondary school probability curriculum. An important way to explore experimental probability and set the stage for a theoretical solution is by using simulations. Turkey Bingo has many possible outcomes; it is difficult to simulate many trials by hand. The software package Fathom Dynamic assists us in finding complex and computation-heavy experimental probabilities, while a spreadsheet (such as Microsoft Excel) is a useful tool for calculating theoretical probabilities. Each will be used in turn.

A Simulation with Fathom

There are many reasons for simulating a probability problem, including these:

- To analyze problems whose theoretical solutions are beyond our current knowledge
- To develop intuition about a complex problem
- To develop intuition that can lead to a theoretical solution

Each of these reasons applies to the Turkey Bingo problem.

Recall that the question at hand asks about 20 winners in a group of 1000 players, a rate of 2 percent. Such a small number will be difficult to detect with a small number of trials. A computer can run a simulation much more quickly than we can draw blocks from a bag or record results from a random number generator. Fathom is computer software designed to do statistical and probability analyses,

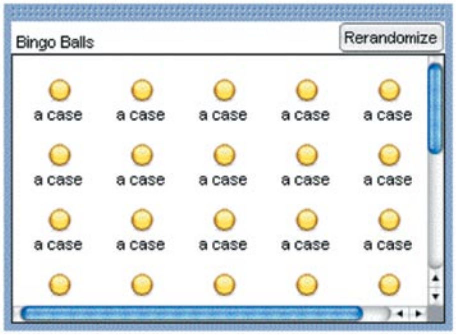

Fig. 3 A collection of cases. Each case is symbolized by a ball. Each case also represents a ball—a bingo ball waiting to be drawn at random.

including sampling and simulation.

The basic unit with which Fathom works is a *case*, symbolized on-screen by a ball (see **fig. 3**). Cases have *attributes*. If we consider a person to be a case, we might be interested in attributes such as height, eye color, IQ, or favorite food. In a probability situation, a case might be a roll of a weighted die. Cases are grouped in *collections*. A collection might be a class full of people or a set of ten rolls of a weighted die. The power of Fathom in creating probability simulations comes in its capability to sample a collection randomly, compute a measure of the sample (e.g., mean height or the sum of ten rolls), and record this measure as a case in a new collection. Each of these capabilities will be used in the Turkey Bingo simulation.

In Turkey Bingo, we may simplify by using numbers (1–40) instead of words, and we consider a case to be a numbered ball waiting to be drawn. The first collection, then, is the set of these 40 balls (see **fig. 4**).

From this collection, we can sample a given number of draws. The original Turkey Bingo game called for sets of 5 words to be drawn at the end of each of four periods—for 20 words altogether—so we take a sample of this size (see **fig. 5**). This sample is a new collection. Fathom allows for sampling with or without replacement. In Turkey Bingo, we never call the same number twice in the same game, so we choose Without Replacement here.

The first analysis is to find the probability that an arbitrary line of 5 words (horizontal, vertical, or diagonal) will win. We may use any line, so consider the combination {1, 2, 3, 4, 5}. (Note that students may see this line as special rather than arbitrary. Discuss this perception and allow students to choose their own arbitrary line, perhaps randomly.) We set up a measure (called row 1) to determine whether this line wins after 20 draws (see **fig. 6**).

The setup so far has required a significant effort, especially in writing the formula in **figure 6** to determine whether the line wins. But now that the

Bingo Balls

	number	word	<new>
1	1	turkey	
2	2	stuffing	
3	3	pilgrim	
4	4	Thanksg…	
5	5	cranberry	
6	6	thanks	
7	7	Lincoln	
8	8	Thursday	

Fig. 4 Eight out of 40 cases in the Bingo Balls collection, represented in table form. Hereafter, only the numbers, not the words, will be considered.

Sample of Bingo Balls

	number	<new>
1	3	
2	9	
3	5	
4	16	
5	29	
6	28	
7	33	
8	12	

Fig. 5 The first 8 cases in a sample of 20

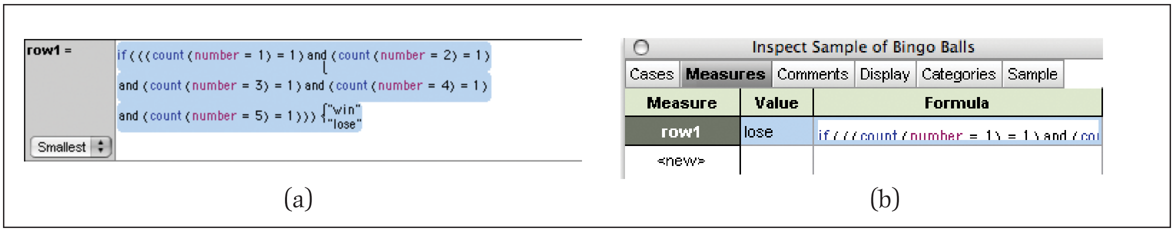

Fig. 6 A formula (**a**) for determining whether the line {1, 2, 3, 4, 5} wins; the result (**b**) (under Value) for the sample partially shown in **figure 5**

setup has been done, the simulation is easy to run. We can quickly generate 1000 samples of 20 draws and see that the experimental probability of this line winning is 2.3 percent (see **fig. 7**). Because there are 12 lines on a card (for simplification, we now consider these independent), we calculate a 27.6 percent chance of any given card winning in 1000 samples of 20 draws. The cards are assumed to be independent of one another, so this probability corresponds to the proportion of 1000 cards that should win on any given trial with 20 draws. The result: 276 winners. Because we have assumed that the lines on each card are independent of one another when in fact they are not, we know this estimate is not quite right (we will analyze the reason for this shortly).

In Fathom, a measure of a collection can become a case in a new collection, allowing us to generate and analyze sampling distributions. This process naturally leads to finding the mean of several collections of 1000 samples of 20 draws. In one such simulation, 5 collections of 1000 samples of 20 had a mean of 15.8 wins. Keeping in mind that each student has 12 lines, we estimate 190 winners on 1000 cards. (At this point, time becomes a factor. On an iMac with an 800 Mhz G4 processor, this simulation [5 sets of 1000 simulations of 20 draws] takes about three minutes.) This number is far above the desired 20 winners, and we can see that the school administration was smart to cut back on the number of words being drawn each period.

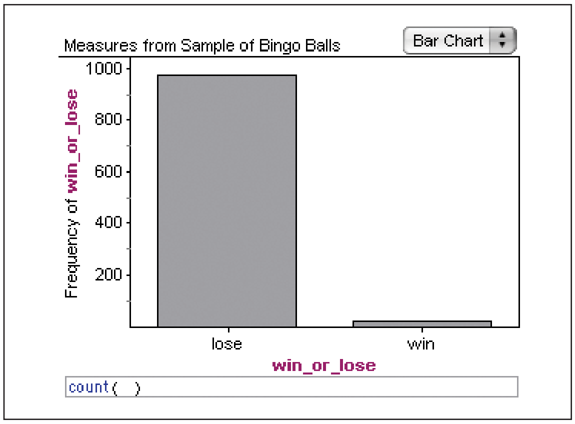

Fig. 7 Bar graph of the number of wins for the line {1, 2, 3, 4, 5} in 1000 trials of 20 draws. The win bar represents 23 wins.

At this stage in the investigation, we encourage students to ask questions that can be answered by modifying the current simulation. Ultimately, we wish to have approximately 20 winners. We could guess-and-check our way by rerunning the simulation with 19 draws, 18 draws, and so on. But if we are going to learn something about probabilities, we need to think about the results of our simulation and plan the next move. Potential questions include these:

- If we cut the number of draws in half (to 10), do we cut the number of winners in half?
- What is the relationship between the number of draws and the number of expected winners?

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Fig. 8 A typical Turkey Bingo card in which numbers have replaced words

- What should we do next to investigate this relationship?

In the end, according to our estimate, drawing a total of 13 words tends to produce about 19 winners. This electronic simulation gives us the power to ask and answer complex questions quickly with experimental data.

We can also build on our previous efforts. The analysis presented here considered the probability of an arbitrary row winning in 1000 draws. We could consider the probability of an arbitrary card winning in 1000 draws and use this simulation to check the accuracy of our estimate. To do so, we set up a separate measure for each line on an arbitrary card. For example, the first column would be {1, 6, 11, 16, 21}, and a diagonal would be {1, 7, 13, 19, 25}; see **figure 8**. A final measure checks whether any of the 12 lines wins and takes on the value "win" if the card contains at least one winning line. With this in hand, we may test the assumption that

$$P(\text{a card winning}) = 12 \cdot P(\text{a line winning}).$$

This assumption turns out to be false. Whether the estimate is close enough depends on one's purposes. A comparison of means of these two measures failed to find a meaningful correlation between the number of estimated wins and the number of actual wins in 50 sets of 1000 trials of 20 draws, although the mean of the estimate was larger than the actual mean (mean estimate wins: 284; mean actual wins: 251) and this difference was statistically significant ($p < .001$).

The suggestion earlier, when we estimated the number of wins, was that 13 words is a good number. Checking this by simulating the number of winning cards in a school of 1000 students 50 times gave a mean of 20.66 and standard deviation of 5.041. Thus, if the desire is to get close to 20 winners, 13 words is a good choice. If it is imperative not to exceed 20 winners, 13 words are too many, and we should choose 12 instead (yielding, in one simulation, a mean of 13.36 and a standard deviation of 3.19).

◇	A	B	C
1	Number of words drawn	Probability of an arbitrary line winning	Number of predicted winning lines schoolwide
2	5	1.51974E-06	0.01823686
3	6	7.59869E-06	0.091184302
4	7	3.79935E-05	0.455921509
5	8	0.00011398	1.367764526
6	9	0.000265954	3.19145056
7	10	0.000531908	6.38290112
8	11	0.000957435	11.48922202
9	12	0.001595725	19.14870336
10	13	0.002507568	30.09081956
11	14	0.003761352	45.13622935
12	15	0.005433065	65.19677572
13	16	0.007606291	91.27548601
14	17	0.010372214	124.4665718
15	18	0.013829619	165.9554291
16	19	0.018084887	217.0186381
17	20	0.023251997	279.0239632
18	21	0.029452529	353.4303534

Fig. 9 Theoretical probabilities computed for number of independent winning lines among 1000 players in a game of Turkey Bingo

Turning from the experimental solution to a computational or theoretical one, we move from Fathom to a spreadsheet: Microsoft Excel.

Analysis with Excel

For this analysis, we also begin with an arbitrary row, say {1, 2, 3, 4, 5}. Following the Bay et al. (2000) analysis, we may ask what the probability is of this line winning in the first 5 draws. For this to happen, the first draw could be any of the 5 numbers in the row, and there are 40 numbers to choose from, so the first draw has a 5/40 chance of being successful. Now there are 4 numbers left in the row and 39 left "in the bag." This leads to the following:

$$P(\text{win on 5 draws}) = 5/40 \cdot 4/39 \cdot 3/38 \cdot 2/37 \cdot 1/36 \approx 1.52 \cdot 10^{-6}$$

We multiply this by 12 to obtain the probability of each student winning to get the value in cell B2 in **figure 9** (we are under the simplifying assumption that the 12 lines are independent of one another) and then by 1000 to obtain the expected number of winning lines in a school of 1000 students (this is the value in cell C2). Note that some of these lines are likely to be held by the same student, who can win only once. Therefore, the number of winning students should be slightly less.

There are 5 places that a nonwinning number could be drawn and still allow our hypothetical line to win in 6 draws. (The nonwinning number could be drawn first, followed by 5 winning draws, and so on, but it could not be drawn sixth.) Therefore, the formula for calculating cell B3 is B2 • 5. Similarly, B4 is B3 • 6/2, dividing by 2 because the order of the extra numbers does not matter. In general, cell Bx is multiplied by

$$\frac{\text{cell A}x}{(\text{cell A}x)-5}.$$

Each time, we multiply by 12,000 to obtain the number of winning lines in the school in column C. From these calculations, Jenson and his students suggested drawing 12 words to obtain about 19 winners; 13 words would have been too many.

Notice that the theoretical estimate closely matches the experimental estimate from Fathom for 20 draws: We calculate about 279 winning lines here, while our simulation produced a mean of 284.16 winning lines over 50 trials. Notice, too, that the reasoning behind the spreadsheet is different from the reasoning behind the simulation. To write the formula for 20 draws, we had to work our way up from 5 draws (why not 1 draw?), whereas the Fathom solution required no such relational thinking. We could arbitrarily choose the number of trials in Fathom. The tendency may be to consider the cases individually.

RESOLUTION

Recall the major question of interest at Jenson's school: How many words should we draw to have the expected number of winners be close to 20 without exceeding that number? The Fathom simulation analyzing the probability of a given card winning gives the answer 12 words, which yields about 13 winners. The computational estimate with Excel (together with the Fathom simulation analyzing the probability of a given row winning) gives the answer 12 words, which yields an expected 19 winners (according to the estimate, 13 words yields 30 winners, which is too large).

Jenson's class suggested that 12 draws be made to yield 19 winners, and this is what the school did. Jenson's results arrived too late to affect the second-period drawing of a single word, but 3 words were drawn at the end of each of the third and fifth periods. Knowing what we know from the Fathom investigation about the error produced by our estimate, we can adjust the spreadsheet predictions to 88 percent of their values (the ratio of the two means of 50 trials). Factoring in attendance on the day before Thanksgiving, we can rerun the numbers for the 930 instead of 1000 students. This yields a final prediction of 15.7 winners on 12 draws and 24.6 winners on 13 draws.

Jenson recounted the results in an e-mail:

> The office decided to go with the lower estimate, reading 3 more words each of the remaining two periods. The actual number of winning students was 12, but since the actual attendance was less than the 1000 we based our predictions on [it was closer to 930], I decided that the actual results closely followed our predicted results.

For teachers interested in using a Turkey Bingo simulation in their own class, step-by-step instructions for using Fathom to simulate the game are available at www.nctm.org/mt10204-248a.

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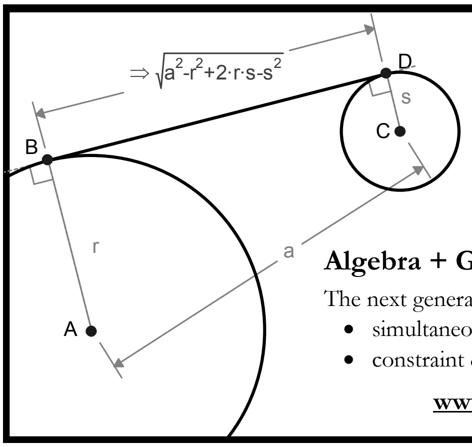

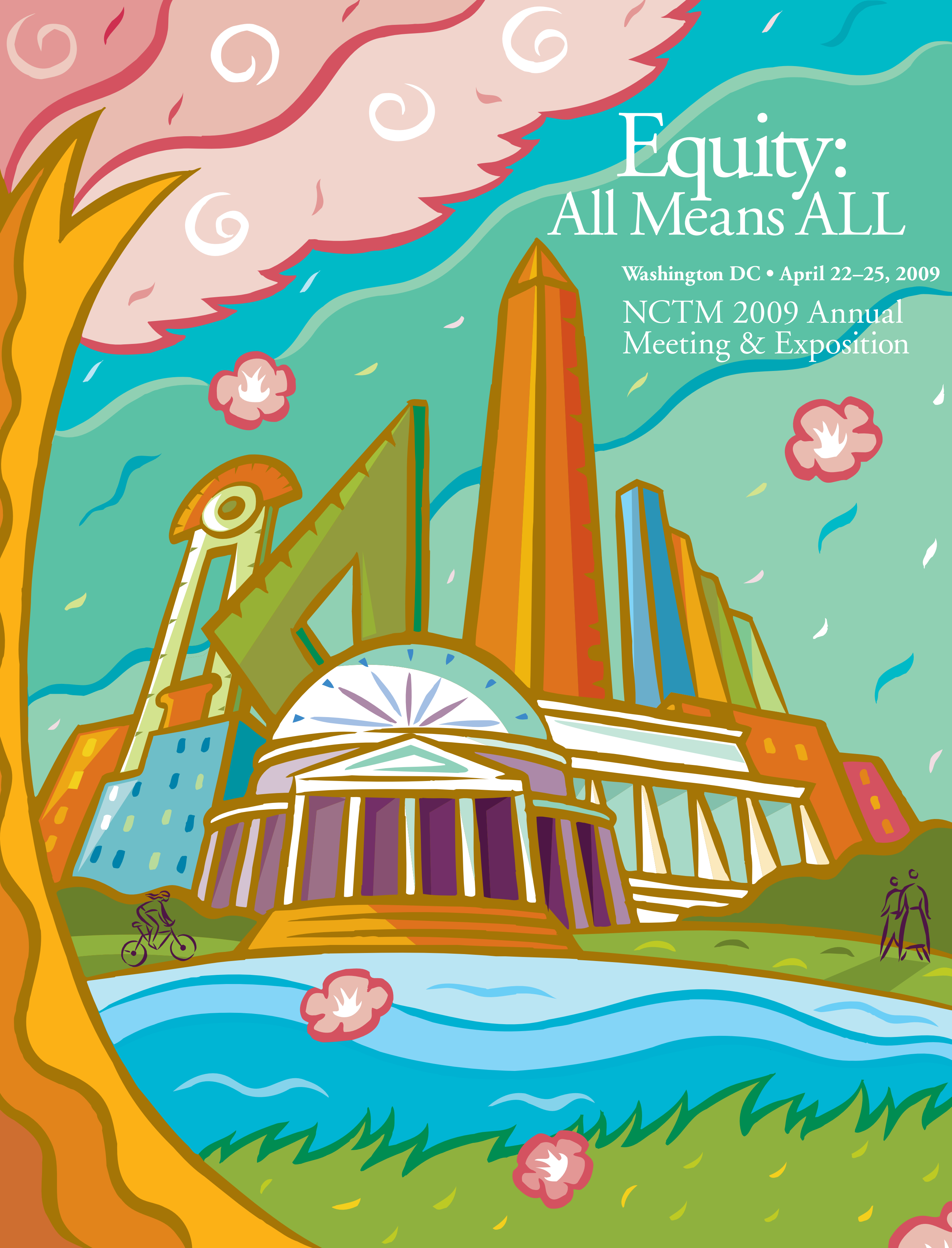

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