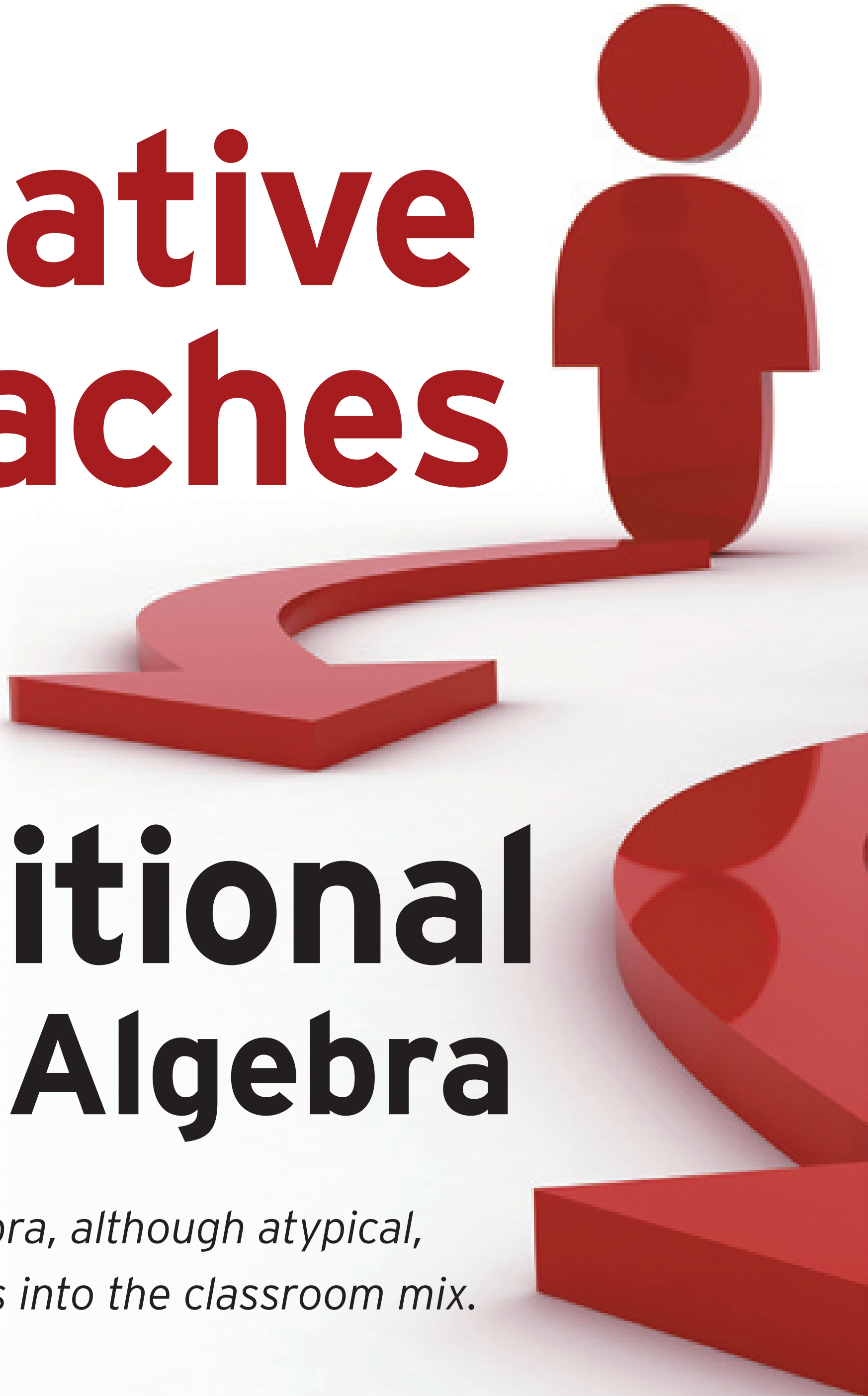
Alternative Approaches to Traditional Topics in Algebra



These new approaches to algebra, although atypical, can bring unreachable students into the classroom mix.

John W. Coburn

Students who otherwise seem unreachable through traditional approaches to algebra require some alternative teaching methods. So do teachers who seek to add elements of freshness and innovation to their classrooms or who simply appreciate variety. This article, as its title indicates, offers some unconventional techniques for teaching a few conventional algebra topics.

We immediately acknowledge two impediments. First, no matter how we change the mathematical landscape, there is no "royal road" to algebra; students will still have to do some form of mathematical thinking. In other words, what makes an alternative process more efficient or easier to understand for some students often introduces mathematical considerations not present in the traditional approach. Second, tradition itself—and perhaps mathematical tradition more so—is hard to break. Even some of the most courageous and forward-thinking

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teachers hold to the tried and true at the expense of innovations that may rightfully have a place in the mathematics curriculum. However, for those with a bold and adventurous spirit, we begin.

TRINOMIAL FACTORING

Many algebra students experience some rough transitions as they move from factoring trinomials of the form $ax^2 + bx + c$ where $a = 1$ to those where $a \neq 1$. The constant term seems to maintain its mesmerizing draw on students' attention, regardless of the lead coefficient and teachers' best efforts. The reasons are perfectly understandable: It is much easier to find two integers that multiply to c and add to b than it is to wade through the vagaries of a trial-and-error process or make an appeal to the formality involved in the ac method. Is there a way to turn this fixation on the constant term into an advantage? There is, particularly when the alternative is viewed in the context of equation solving.

Consider the equation $6x^2 - 7x - 3 = 0$. A student looking for two integers whose product is –3 and whose sum is –7 may quickly say that the polynomial is prime because the only factors of c are 1 and –3 or –1 and 3. An alternative approach, however, is to divide through by 6, rewriting the equation as

$$x^2 - \frac{7}{6}x - \frac{3}{6} = 0.$$

At first glance, the introduction of rational coefficients seems to exacerbate the situation, unless a student is willing to accept that adding fractions with like denominators is a triviality.

The alternative approach uses the fact that if a sum of two fractions must be expressed in sixths ($b = -7/6$), the product should be expressed in thirty-sixths for ease of computation: $c = -3/6 = -18/36$. With the equation now written as

$$x^2 - \frac{7}{6}x - \frac{18}{36} = 0,$$

we find the factored form by seeking two integers whose sum is –7 and whose product is –18—namely, –9 and 2. The factored form is then $(x - 9/6)(x + 2/6) = 0$, with solutions $x = 3/2$ or $x = -1/3$ (after simplifying). Naturally, the value of this approach will be directly proportional to a student's capacity (or willingness) to overcome an aversion to rational numbers.

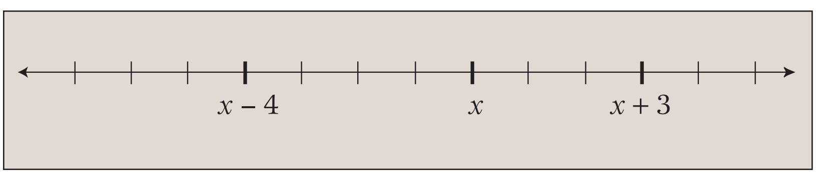

Fig. 1 Taking the factors in increasing order can make using the push principle easier.

Example 1: Solve $2x^2 + 7x - 15 = 0$.

Solution:

$$\begin{aligned} 2x^2 + 7x - 15 &= 0 \\ x^2 + \frac{7}{2}x - \frac{15}{2} &= 0 \\ x^2 + \frac{7}{2}x - \frac{30}{4} &= 0 \\ \left(x + \frac{10}{2}\right)\left(x - \frac{3}{2}\right) &= 0 \end{aligned}$$

The solutions are $x = -5$ or $x = 3/2$.

SOLVING INEQUALITIES USING THE PUSH PRINCIPLE

The most common method of solving polynomial inequalities such as $x^2 - x - 12 > 0$ involves finding the zeroes of the function and checking the sign of the function in the intervals between these zeroes. Students who are "graphically inclined" rely on the multiplicity of each zero and either the end behavior or the sign of the function at the y-intercept; they can then draw a rough sketch of the graph from which the solution can be read.

A third method is more conceptual in nature but in many cases highly efficient. It is based on two very simple ideas, the first involving only order relations and the number line:

Given any number x and constant $k > 0$,

$$x > x - k \text{ and } x < x + k. \qquad (1)$$

This statement simply reinforces the idea that if a is to the left of b on the number line, then $a < b$. As shown in **figure 1**, $x - 4 < x$ and $x < x + 3$, yielding $x - 4 < x + 3$ for any x.

The second idea reiterates well-known concepts regarding the multiplication of signed numbers:

> For any number of factors, if there is an even number of negative factors, the result is positive; if there is an odd number of negative factors, the result is negative. (2)

These two ideas work together to solve inequalities using what we will call the *push principle*. Consider the inequality $x^2 - x - 12 > 0$, with factored form $(x - 4)(x + 3) > 0$. From (1), we know that both factors will be positive if $(x - 4)$ is positive, because it is the smaller factor ($x - 4$, as positive, "pushes" $x + 3$ to be positive). Moreover, both factors will be negative if $x + 3 < 0$ because it is the larger factor ($x + 3$, as negative, "pushes" $x - 4$ to be negative). We find the solution set by solving these two simple inequalities; the result is $x > 4$ or $x < -3$.

If the original inequality were $(x - 4)(x + 3) < 0$ instead, we require one negative and one positive


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factor. Order relations and the number line require that the larger factor be the positive one—$x + 3 > 0$, so $x > -3$—and that the smaller factor be the negative one—$x - 4 < 0$, so $x < 4$. The solution is $-3 < x < 4$, as we can verify using any alternative method. The solutions to all other polynomial and rational inequalities are an extension of these two cases.

Example 2: Solve $x^3 - 7x + 6 \le 0$ using the push principle.

Solution: We can factor this polynomial by noting that $x = 1$ is a root and using synthetic division. The factored form is $(x - 2)(x - 1)(x + 3) < 0$, which we have conveniently written with the factors in increasing order. From (2), we know that for the product of three factors to be negative we require three negative factors or one negative and two positive factors. The first condition is met by simply making the largest factor negative, thus ensuring that the smaller factors are also negative: $x + 3 < 0$, so $x < -3$. The second condition is met by making the smallest factor negative and the "middle" factor positive: $x - 2 < 0$ and $x - 1 > 0$, yielding $1 < x < 2$. The complete solution is $x \in (\infty, -3] \cup [1, 2]$.

Note that the push principle does not require testing intervals between the zeroes or analyzing whether the graph crosses or bounces off the x–axis at zeroes and vertical asymptotes (of rational functions) that would be necessary if we were using a graphical approach. As an additional benefit, we can ignore irreducible quadratic factors (they contribute nothing to the solution set because they yield complex zeroes) as well as factors of even multiplicity (there is no sign change at the related root).

Example 3: Solve $(x^2 + 1)(x - 2)^2(x + 3) > 0$ using the push principle.

Solution: The factor $(x^2 + 1)$ does not affect the solution set (the factor produces no real zeroes and hence no sign changes), so this inequality will have the same solution as $(x - 2)^2(x + 3) > 0$. Further, $(x - 2)^2$ will be nonnegative for all x, so the inequality has the same solution set as $(x + 3) > 0$. The solution is $x > -3$.

Example 4: Solve the inequality

$$\frac{x^3 - x^2 + 4x - 4}{x^3 - 3x^2 - 9x + 27} < 0.$$

Solution: In factored form, the inequality is

$$\frac{(x^2 + 4)(x - 1)}{(x - 3)^2(x + 3)} < 0.$$

Neither of the factors $(x^2 + 4)$ or $(x - 3)^2$ will ever be nonnegative when the expression is defined and can be ignored, indicating that the original inequality has the same solution set as

$$\frac{(x-1)}{(x+3)} < 0, x \neq -3.$$

From (2), we know that the ratio of two factors will be negative when we have one negative factor and one positive one, giving the solution $x - 1 < 0$ and $x + 3 > 0$ (the smaller factor must be negative and the larger factor positive). The solution is $-3 < x < 1$.

GRAPHING AND SOLVING QUADRATIC FUNCTIONS

Certain transformations of quadratic graphs offer an intriguing alternative to graphing these functions by completing the square. In many cases, the new process is less time-consuming and ties together a number of important concepts. To begin, for the function $f(x) = ax^2 + bx + c$, we will call $F(x) = ax^2 + bx$ the base function—that is, the original function less the constant term.

When we compare $f(x)$ with $F(x)$ (see **fig. 2**), we notice the following. First, F and f share the same axis of symmetry (one is simply a vertical shift of the other). Second, we can easily find the x-intercepts of F by factoring (because $0 = ax^2 + bx$ gives $0 = ax(x + b/a)$ with solutions $x = 0$ and $x = -b/a$). Third, the axis of symmetry h is simply the average value of the x-intercepts:

$$h = \frac{x_1 + x_2}{2} = \frac{-b}{2a}$$

Fourth, the vertices of F and f differ only by the constant c. If we consider these vertices to be (h, k_0) and (h, k), respectively, we have $k = k_0 + c$.

In what follows, we discover an additional insight that makes the graphing of many quadratics swift and efficient. To begin, we will find k_0 by evaluating F at $-b/2a$. We start with $F(x) = ax^2 + bx$ and then substitute $-b/2a$ for x:

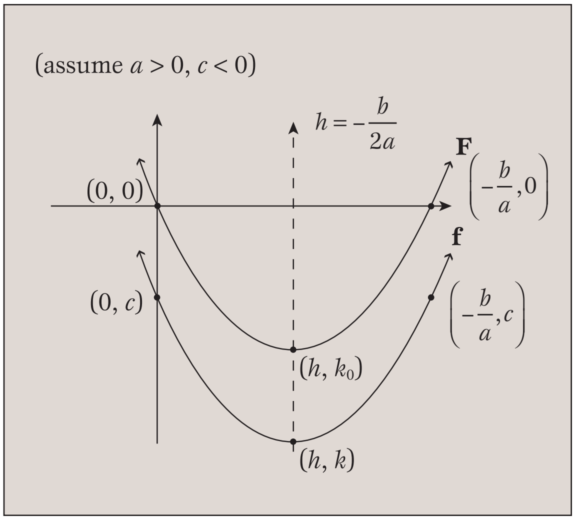

Fig. 2 It is much easier to work first with F than with f.

$$\begin{aligned}F\left(\frac{-b}{2a}\right) &= a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right)\\ &= a\left(\frac{b^2}{4a^2}\right) - \frac{b^2}{2a}\\ &= a\left(\frac{b^2}{4a^2}\right) - \frac{b^2}{2a} \bullet \frac{2a}{2a}\\ &= a\left(\frac{b^2}{4a^2}\right) - 2a\left(\frac{b^2}{4a^2}\right)\\ &= -a\left(\frac{b}{2a}\right)^2\end{aligned}$$

From $h = -b/2a$, we have $-h = b/2a$. It follows that

$$\begin{aligned}F\left(\frac{-b}{2a}\right) &= -a\left(-h\right)^2\\ &= -ah^2\end{aligned}$$

This verifies that the vertex of F is (h, k_0) where $h = -b/2a$ and $k_0 = -ah^2$.

Note that we can now determine the vertex of both F and f using only elementary operations on the single value h because $k_0 = -ah^2$ and $k = k_0 + c$. In addition, because the vertex of f is known, we can find the zeroes using a vertex-intercept formula (instead of requiring the quadratic formula). From $f(x) = a(x - h)^2 + k$, we can easily find the zeroes of f:

$$x = h \pm \sqrt{-\frac{k}{a}}$$

As a bonus, this approach allows quick access to the exact form of the roots, even when they happen to be irrational or complex.

Following are two more examples solved by using the new approach.

Example 5: Graph the function $f(x) = x^2 - 10x + 17$ and locate its zeroes (if they exist).

Solution: For $F(x) = x^2 - 10x$, the zeroes are 0 and 10 by inspection, with x-intercepts (0, 0) and (10, 0) and $h = 5$ (the halfway point) as the axis of symmetry. The vertex of F is $(h, -ah^2)$ or (5, –25). With $c = 17$, we add 17 units to the y-coordinates of these three points and find that the graph of f will contain (0, 17) [the y-intercept], (10, 17), and (5, –8) [the vertex]. Because $a = 1$, the zeroes of f are $(h \pm \sqrt{k}, 0) = (5 \pm \sqrt{8}, 0)$, or approximately (2.2, 0) and (7.8, 0).

If b is an odd number, the decimal form of the halfway point can be used to help locate the vertex. One needs to realize that the square of any number ending in 5 is the product of the preceding digit or digits and the next larger integer, with 25 appended. For instance, if $b = 7$, $7/2 = 3.5$, and $(3.5)^2 = 3(3 + 1) + 0.25$, or 12.25.

Even when $a \neq 1$, the alternative method lends a measure of efficiency to graphing and solving quadratic functions, as shown in example 6.

Example 6: Graph the function $f(x) = -2x^2 + 5x - 4$ and locate its zeroes (if they exist).

Solution: For $F(x) = -2x^2 + 5x$, the zeroes or x-intercepts are (0, 0) and (5/2, 0) by inspection, with $h = 5/4$ (the halfway point) as the axis of symmetry. Noting that $a = -2$ and $c = -4$, we find that the vertex of F is at (5/4, 25/8) because

$$\begin{aligned} -ah^2 &= 2\left(\frac{5}{4}\right)^2 \\ &= \frac{25}{8}. \end{aligned}$$

After subtracting $4 = 32/8$ units from the y-coordinates of these three points, we find that the graph of f will contain (0, –4) [the y-intercept] and (5/2, –4) with vertex (5/4, –7/8). The roots of f will be

$$\begin{aligned} x &= \frac{5}{4} \pm \sqrt{\frac{-7}{16}} \\ &= \frac{5}{4} \pm \frac{\sqrt{7}}{4} i, \end{aligned}$$

indicating that the graph has no x-intercepts.

CONCLUSION

These approaches are atypical (at least in my experience). However, they seem to have a certain appeal to many students, particularly those who have already seen more typical ways to do the work. Overall, student responses have been quite positive. Even more notable, I have seen these approaches embraced by students of all ability levels, not only those whom we might consider the brightest or hardest working. I encourage you to give these new ideas a try.

For additional problems for students to solve and graph, go to the *Mathematics Teacher* Web site: www.nctm.org/mt.

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ADDITIONAL PROBLEMS

Here are a few exercises to try with your students.

Solve the following:

1. $x^3 - 3x - 18 < 0$
2. $\dfrac{(x+1)(x^2+1)}{x^2-4} > 0$
3. $x^3 - 13x + 12 < 0$
4. $x^3 - 3x + 2 > 0$
5. $x^4 - x^2 - 12 > 0$
6. $(x^2+5)(x^2-9)(x+2)^2(x-1) > 0$

Graph the following:

1. $f(x) = x^2 + 2x - 7$
2. $g(x) = x^2 + 5x + 9$
3. $h(x) = x^2 - 6x + 11$
4. $p(x) = -x^2 + 10x - 17$
5. $q(x) = 2x^2 + 12x + 21$
6. $r(x) = 2x^2 - 7x + 8$