AUGUST

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

This month's "Calendar" is a relay calendar: In each row of the calendar, the answer to a problem will be used in the next consecutive question. The answer to problem n will be represented by A_n. This relay format is used in many competitions, including those of the American Regions Mathematics League.

On a reality show, teams of 6 players have to work together to accomplish a task. Each member of the team is tied to each of his or her teammates with a separate strand of rope. How many pieces of rope are required to tie together all 6 members of the team?

1

While sitting in his car at a train crossing, the driver decided to time how long it took for a 1-mile-long train to pass him entirely. If it took exactly $5 \cdot A_1$ seconds for the train to pass, at how many miles per hour was the train traveling?

2

In a local hotel there are A_2 steps from the first floor to the fourth floor. If we assume that the number of steps between consecutive floors is constant and that it takes 1.5 seconds to climb one step, how many seconds will it take to climb the stairs from the first floor to the tenth floor?

3

The sum of two integers is 28, and their product is 192. What are the two integers?

4

Let the two results called A_4 be the lengths of the diagonals of a rhombus. Determine the length of the radius of the circle inscribed in the rhombus.

5

Let $5 \cdot A_5$ be the measure of side ST in triangle $\triangle RST$ with $m\angle T = 30°$. Determine the number of integral values for side RS that will result in two distinct values for length RT.

6

Let $\log_k 2 = A_6/9$. Find $\log_8 k$.

7

Find the sum of the odd integers from 13 to 107, inclusive.

8

Determine the number of sides that a polygon must have if the sum of its interior angles, measured in degrees, is A_8.

9

The longest side of a triangle is A_9, and the shortest side is $A_9/2$. If the angle between the two known sides is 60°, find the length of the triangle's third side.

10

The numbers a and b are positive integers with $a < b$. Solve for a and b in the following equation:

$$a^3 + b^3 = A_{10}{}^2$$

11

Driving west along U.S. Route 66, Kevin saw a sign that read: Amboy 128 miles, Needles 202 miles. Later, a second sign indicated that the distance to Needles was twice the distance to Amboy. Later still, a third sign indicated that the distance to Needles was three times the distance to Amboy. How many miles apart were the second and third signs?

12

Let A_{12} be the hypotenuse of a right triangle with integer legs. Find the positive difference between the two legs of the triangle.

13

Two streets mutually terminate, forming an angle of 52°. A tree is to be planted so that it is equidistant from the two streets and A_{13} ft. from the intersection. Find, to the nearest integer, the number of feet the tree will be from each street.

14

A game is played with a square board that is A_{14} ft. on a side. Pegs are placed at the four corners of the board. A ring whose radius is 1 ft. is tossed onto the board. If we assume that the center of the ring falls somewhere on the board, what is the probability that it encircles one of the pegs?

15

How many cubes, each 3 in. on a side, can fit in a rectangular prism whose dimensions are 2 ft. × 3 ft. × 4 ft.?

16

Let $p = A_{16}/100$ rounded to the nearest integer. Let $q = A_{16} - 100p$. The Yankees won p games out of the first q games played. What is the minimum number of additional games they would have to play to win 3/4 of the total number of games played?

17

In isosceles triangle DEF, $DE = EF = A_{17}$. Altitudes from E and F are drawn and intersect the opposite sides of the triangle at G and H, respectively. If $DH = 6$, find DF.

18

Determine the number of sides of a convex polygon that contains exactly $3.75 \cdot A_{18}$ diagonals.

19

What is the smallest integer that is divisible by all the integers 1 through 10, inclusive?

20

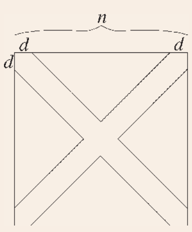
Let s = the sum of the digits in A_{20}, and let p = the product of the nonzero digits of A_{20}. Suppose that we have points J, K, L, and M on a circle with $JK = s$, $JM = p$, and the diameter $JL = s + p$. Perpendicular segments are drawn to JL from both K and M and intersect JL at E and F, respectively. Find EF.

21

Triangle $\triangle RED$ has point G on side RD. If $RE = 12$, $RG = 9$, $GD = 7$, and the perimeter of $\triangle EGD = 3A_{21} + 2$, determine the perimeter of $\triangle REG$.

22

Factor A_{22} into primes. Let n be the largest prime factor and d the smallest. A square is decorated with a symmetric X shape. If a side of the square is n and d is as needed, determine the area of the X.



23

It takes a person 6 days to cross a stretch of deserted land. One person can carry only enough food and water to survive for 4 days. What is the minimum number of people who must start out to ensure that one person successfully gets across and that the others safely return to the starting point?

24

Bonnie has coins that total more than $A_{24}/3$ dollars. Her friend asks her if she has change for a dollar bill and is surprised that she does not. What is the largest amount of money, in cents, that Bonnie can have and still not be able to give her friend change for a dollar?

25

The area of the base of a right rectangular pyramid is $100 \cdot A_{25} - 11$. If the lateral areas of two adjacent faces are 45 and $3\sqrt{97}$, compute the volume of the pyramid.

26

Suppose that for all $x \geq 0$,

$$f(g(x)) = \sqrt{x},\ g(f(x)) = x^2, \text{ and } g(12) = 25.$$

Find $g(A_{26})$.

27

When the waiter was asked to slice the 16-in. (in diameter) pizza into exactly 4 slices of equal size, he decided to do so using concentric circles rather than radial slices. Determine the radii of each of the three cuts he had to make.

28

Find the number of square units in the area of a triangle whose sides measure the three values of A_{28}.

29

Determine $17 \cdot A_{29}{}^2$ and arrange the resulting digits in decreasing order. Then arrange the same digits in increasing order and subtract the smaller number from the larger. Repeat the process with the current result. Determine the number of times this process must be performed before you arrive at the number 6174 for the first time.

30

Given that

$$x^2 + 1/x^2 = A_{30},$$

evaluate the absolute value of

$$x^2 - 1/x^2.$$

31

SOLUTIONS to calendar

Looking for more "Calendar" problems?
Visit www.nctm.org/publications/mtcalendar for a collection of previously published problems—sortable by topic—from the *Mathematics Teacher.*

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Problems 1–23 and 27–29 have been adapted from the Upper School Mathematics Competitions conducted by the Independent Schools Mathematical Association of Washington (D.C.). Problems 24–26 have been adapted from the New York City Interscholastic Mathematics League.

The Editorial Panel of the *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly "Calendar." Send problems to the "Calendar" editors. Remember to include a complete solution for each problem submitted.

Other sources of problems in calendar form available from NCTM include *Calendar Problems from the "Mathematics Teacher"* (a book featuring more than 400 problems, organized by topic; stock number 12509, $22.95) and the 100 Problem Poster (stock number 13207, $9.00). Individual members receive a 20 percent discount off this price. A catalog of educational materials is available at www.nctm.org.—Eds.

1. 15. Each person must be tied to his or her 5 teammates. Since 6 people are each tied to 5 others and we do not wish to count a rope twice, the number of pieces of rope needed will be $6 \cdot 5 \div 2 = 15$.

Alternate solution: Person P is tied to 5 other people. Person Q is tied to 4 additional teammates. Person R needs only 3 additional strands of rope. Continuing in this fashion, we find that the total number of ropes is $5 + 4 + 3 + 2 + 1 = 15$.

2. 48. Convert $5A_1 = 75$ seconds into 1.25 minutes and then divide 60 minutes by 1.25 minutes to get 48.

Alternate solution: Convert miles per second into miles per hour by dividing 3600 by 75 to get 48 mph. Unit analysis offers another approach:

$$\frac{1\text{ mile}}{75\text{ sec}} \cdot \frac{60\text{ sec}}{1\text{min}} \cdot \frac{60\text{ min}}{1\text{ hr}} \rightarrow$$

$$\frac{1\text{ mile}}{75\,\cancel{\text{sec}}} \cdot \frac{60\,\cancel{\text{sec}}}{1\cancel{\text{min}}} \cdot \frac{60\,\cancel{\text{min}}}{1\text{ hr}} = \frac{3600\text{ mi}}{75\text{ hr}}$$

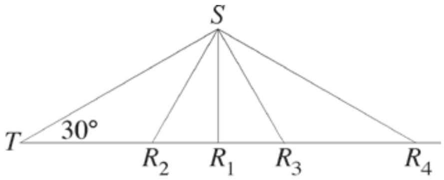
$$= 48\text{ mi/hr}$$

3. 216 seconds. Since there are 3 floors from the first to the fourth, there are 16 steps between floors. Each floor requires 24 seconds, and we wish to climb all 9 flights of stairs. The time required is thus $9 \cdot 24$, or 216, seconds.

4. 12 and 16. If we let one integer be a, the other is $28 - a$. Then $a(28 - a) = 192$, and we can obtain a quadratic equation in standard form: $a^2 - 28a + 192 = 0$. Factor the left side of this equation to get $(a - 12) \cdot (a - 16) = 0 \rightarrow a = 12$ or $a = 16$. Students who immediately recognize the problem as a quadratic relation may recall that, for roots r_1 and r_2, $x^2 - (r_1 + r_2)x + (r_1r_2) = 0$. So, again, $x^2 - 28x + 192 = 0$.

5. 4.8. We present three different solutions, each of which depends on the fact that the radius of the inscribed circle is half the altitude of the rhombus. In solution 1, the diagonals of a rhombus are perpendicular bisectors of each other, so the four right triangles they form are 6-8-10 triangles. Thus, the length of a side of the rhombus is 10. The radius of the inscribed circle is the altitude to the hypotenuse of any of these four triangles and divides the triangle into similar triangles. Form the proportion $r/6 = 8/10 \rightarrow r = 4.8$. In solution 2, we calculate the area of a rhombus in two ways: $A = bh = 0.5(d_1)(d_2)$. Replace b, d_1, and d_2 with 10, 12, and 16, respectively, and solve for h, the diameter of the circle. In solution 3, solve for the area of one of the four triangles in two ways: $A = 0.5 \cdot 6 \cdot 8 = 0.5 \cdot h \cdot 10$ and solve for h, the height of the triangle and also the radius of the circle.

6. 11. $5A_5 = 24$, and the altitude from S is half that, or 12, as shown in the figure (next column, top). The ambiguous case for the law of sines gives us $12 < R_kS < 24$. There are 11 such integers.



7. 3/11. Use the change-of-base formula to write the following:

$$\log_8 k = \frac{\log_k k}{\log_k 8} = \frac{\log_k k}{\log_k 2^3}$$
$$= \frac{\log_k k}{3\log_k 2} = \frac{1}{3 \cdot \frac{11}{9}} = \frac{3}{11}$$

Alternate solution 1: The same formula can be used as follows:

$$\log_k 2 = \frac{\log_8 2}{\log_8 k} = \frac{1}{3\log_8 k} = \frac{11}{9} \to$$
$$\log_8 k = \frac{3}{11}.$$

Alternate solution 2: Changing to exponential form, we obtain the following: $\log_k 2 = 11/9 \to k^{11/9} = 2 \to (k^{11/9})^{9/11} = 2^{9/11}$. We have $2^{9/11} = (2^3)^{3/11} = 8^{3/11}$. Since $k = 8^{3/11}$, $\log_8 k = 3/11$.

8. 2880. Determine the number of terms in the series: $(107 - 13)/2 + 1 = 48$. The average value in the set of addends is $(107 + 13)/2 = 60$, and $60 \times 48 = 2880$.

Alternate solution 1: There are 6 odd integers from 1 to 11, so the number of odd integers from 13 to 107 is $(1/2) \cdot (108) - 6 = 48$.

Alternate solution 2: Notice that the sum of the first and last terms is the same as the sum of the second and next-to-last terms. This pattern will continue for all the corresponding pairs. Multiply this sum, 120, by the number of pairs of terms, 24, to get 2880.

9. 18. The sum of the interior angles of an n-sided polygon is $180(n - 2) = 2880$, so $n = 18$.

10. $\sqrt{243} = 9\sqrt{3}$. Since we are given two sides and an included angle, this problem can be solved by a straightforward application of the law of cosines: $x^2 = 18^2 + 9^2 - 2(18)(9)\cos(60°) = 243$. However, we do have SAS and therefore know that there is a unique triangle, that the triangle has a 60° angle, and that the side opposite the smallest angle is half the side opposite the largest angle. Thus, we can conclude that we have a special right triangle and that the side opposite the 60° angle is half the hypotenuse times the square root of 3.

11. 3 and 6. Since $a^3 + b^3 = 243 = 3^5$ and $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, we know that $a + b$ must be an integral power of 3 less than 243. Therefore, $a + b = 3, 9, 27$, or 81. The set of perfect cubes less than 243 contains 1, 8, 27, 64, 125, and 216. Consequently, the values for a and b come from the set $\{1, 2, 3, 4, 5, 6\}$. Combining these two ideas, we find that the only solution is $a = 3$ and $b = 6$.

12. 37. Solve the equation $202 - x = 2(128 - x)$ to find that the distance from the first sign to the second sign is $x = 54$ miles. Solve the equation $202 - y = 3(128 - y)$ to find that the distance from the first sign to the third sign is $y = 91$ miles. Subtract the two values to find the distance from the second to the third sign: $91 - 54 = 37$.

13. 23. We search systematically for integer solutions by subtracting perfect squares from $37^2 = 1369$. Only two differences are perfect squares: $37^2 - 12^2 = 1369 - 144 = 35^2$ and $37^2 - 35^2 = 1369 - 1225 = 12^2$. The positive difference is $35 - 12 = 23$.

Alternate solution: One family of Pythagorean triples is $2n$, $n^2 - 1$, and $n^2 + 1$. If we let $n^2 + 1 = 37$, then $n = 6$, and we see that 12 and 35 are the lengths of the two legs.

14. 10. The locus of points equidistant from the sides of an angle is the angle bisector, while the points 23 ft. from a given point form a circle of radius 23 ft. These two loci intersect at one point. If d is the distance from the tree to either street, then $\sin(26°) = d/23 \to d = 23\sin(26°) \to d \approx 10.1$ ft., or, to the nearest integer, 10 ft.

15. $\pi/100$. To encircle a peg, the center must be within one of the quarter-circles of radius 1 ft. surrounding a peg. Let the center of the ring be called P. Since the area of the board is $A_{14}{}^2 = 100$ sq. ft. and the area of each of the four quarter-circles surrounding a peg is $\pi/4$, the probability that P falls in one of the quarter-circles is $\pi/100$.

16. 1536. 2 ft. × 3 ft. × 4 ft. is equivalent to 24 in. × 36 in. × 48 in. Therefore, there are 8 3-in. cubes along the first edge, 12 along the second edge, and 16 along the third edge. Hence, there will be $8 \times 12 \times 16 = 1536$ small cubes in the larger prism.

17. 48. First, determine that $p = 15$ and $q = 36$. Then solve the proportion

$$\frac{15 + x}{36 + x} = \frac{3}{4}$$

to get $x = 48$.

18. 24. Use the fact that $DG = DF/2$ and that $\triangle DHF \sim \triangle DGE$. Set up and solve the proportion $48/x = 2x/6 \to 2x^2 = 288 \to x^2 = 144 \to x = 12$, so $2x = 24$.

***Alternate solution*:** If $DH = 6$, then $HE = 42$. Use the Pythagorean theorem first in $\triangle EFH$ to determine that $FH^2 = 540$ and then in $\triangle DHF$ to find DF.

19. 15. Each diagonal connects two vertices, and no diagonals are drawn to neighboring vertices or to the vertex itself. The number of diagonals meeting at a vertex is the same for all vertices and equals 3 less than the number of vertices. Then, $A_{18} \cdot 3.75 = 24 \cdot 3.75 = 90$ diagonals. If we let n be the number of the polygon's sides, which is the same as the number of its vertices, then $n(n - 3)/2 = 90$. Solving for n yields 15.

20. 2520. We obtain 2520 by multiplying $2^3 \cdot 3^2 \cdot 5 \cdot 7$, the factors of the divisors. Working backward from 10 to build the answer, we need to include 2 and 5 so that 10 will be a divisor, 3^2 so that 9 will be a divisor, and 2^3 so that 8 will be a divisor. Now, all the numbers less than 8 except 7 will be factors of our number. Therefore, we need to include only 7.

21. 11. Draw $\overline{KL}$ and $\overline{LM}$ as shown in the figure. Then $\triangle JMF \sim \triangle JLM$ and $\triangle JKE \sim \triangle JLK$ since angles inscribed in semicircles are right angles. Then

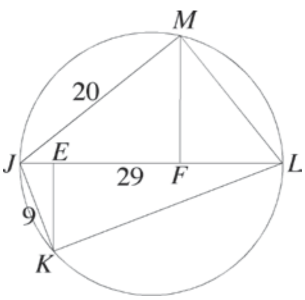
$JF/JM = JM/JL \rightarrow JF/20 = 20/29 \rightarrow JF = 400/29$

and

$JE/JK = JK/JL \rightarrow JE/9 = 9/29 \rightarrow JE = 81/29.$

Finally,

$EF = JF - JE = 400/29 - 81/29 = 319/29 = 11.$



***Alternate solution*:** After drawing the auxiliary lines, use the theorem that states that when an altitude is drawn to the hypotenuse of a right triangle, either leg is the mean proportional between the whole hypotenuse and the segment nearer the leg to obtain the same equations.

22. 33. The key here is to realize that $\triangle REG \sim \triangle RDE$ by the SAS similarity theorem (i.e., $9/12 = 12/16$, so $RG/RE = RE/RD$). It follows that their perimeters will have the same ratio: 3/4. Let $EG = x$, so $ED = 4x/3$. Since the perimeter of $\triangle EGD = 3A_{21} + 2 = 3(11) + 2 = 35$, we can find x by solving the equation $x + 4x/3 + 7 = 35$. Therefore, $x = 12$, and the perimeter of $\triangle REG = 33$.

23. 96. The area of one of the side triangles can be found by using this formula: Area $= 0.5s^2$ where s is a side of the right isosceles triangle. The hypotenuse of the triangle is $n - 2d = 5$, so $s = 2.5\sqrt{2}$ and the area is 25/4. Since there are four such triangles, we can subtract the sum of their areas from the area of the square to find the area of the X: $121 - 25 = 96$.

Alternate solution: The four right isosceles triangles form a square with side length $n - 2d$. Subtracting $(n - 2d)^2$ from n^2 results in the requested area. Since $n = 11$ and $d = 3$, the area of the X shape is $11^2 - 5^2 = 96$.

24. 3. If 3 people start out, at the end of the first day one person gives each of the remaining people a 1-day supply of food and water and then returns to the starting position. At the end of the second day, 1 of the 2 remaining people gives the other person enough food and water for 1 day and then returns to the starting position. The remaining person has 4 days' worth of supplies and can complete the 4-day trek.

25. \$1.19. This solution can be determined by using either 1 quarter, 9 dimes, and 4 pennies or 3 quarters, 4 dimes, and 4 pennies.

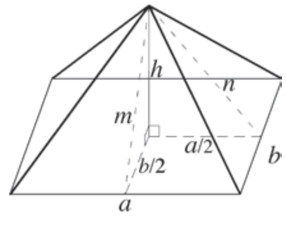
26. 144. Label the pyramid as shown in the figure. The volume of the pyramid is $V_{\text{pyramid}} = (A_{\text{base}} \cdot h)/3 = 108h/3 = 36h$. To calculate h, we need to solve the system of equations:

$$ab = 108$$
$$45 = am/2 \rightarrow am = 90$$
$$3\sqrt{97} = bn/2 \rightarrow bn = 6\sqrt{97}$$
$$h^2 = n^2 - (a/2)^2$$
$$h^2 = m^2 - (b/2)^2$$

Since $m = 90/a$ and $n = 6\sqrt{97}/b$, we can create two equations in two variables:

$$h^2 = \left(\frac{6\sqrt{97}}{b}\right)^2 - \left(\frac{108}{2b}\right)^2$$
$$h^2 = \left(\frac{90b}{108}\right)^2 - \left(\frac{b}{2}\right)^2$$

Multiply and simplify to get $h^2 = 576/b^2$ and $h^2 = 4b^2/9$. Equating these results yields $b = 6$ and $h = 4$. Thus, $V_{\text{pyramid}} = 36h = 36(4) = 144$.



27. 5. Examine the expression $g(f(g(144)))$ in two different ways. First consider $g[f(g(144))] = g(\sqrt{144}) = g[12] = 25$. Then consider $g(f[g(144)]) = [g(144)]^2$. Thus, $[g(144)]^2 = 25$, so $g(144) = 5$.

28. 4, $4\sqrt{2}$, and $4\sqrt{3}$. The area of the whole pizza is $8^2\pi = 64\pi$, so each slice must be 16π. The innermost radius must be 4 in. The radius of the next slice, in the shape of a ring, can be calculated using $\pi r_2^2 = 32\pi$ (basically, 2 slices of the pie), so $r_2 = \sqrt{32} = 4\sqrt{2}$. The third and final radius can be found in a similar fashion: $\pi r_3^2 = 48\pi$ (3 slices), so $r_3 = \sqrt{48} = 4\sqrt{3}$.

29. $8\sqrt{2}$. The triangle with sides 4, $4\sqrt{2}$, and $4\sqrt{3}$ is a right triangle since the values satisfy the Pythagorean theorem. The area of the triangle $= 0.5(4)(4\sqrt{2}) = 8\sqrt{2}$.

30. 4. The first iteration is $17(8\sqrt{2})^2 = 2176$ and $7621 - 1267 = 6354$; the second is $6543 - 3456 = 3087$; the third is $8730 - 0378 = 8352$; and the fourth is $8532 - 2358 = 6174$. If you choose any 4-digit number not divisible by 1111, the process described here will always lead to the result 6174. Since $7641 - 1467 = 6174$, the process will continue to produce 6174. This statement has been verified for all 4-digit numbers that are not multiples of 1111, but no mathematical reason has been found to explain this phenomenon.

31. $\sqrt{12} = 2\sqrt{3}$. We have $x^2 + 1/x^2 = 4$, so, by squaring both sides of the equation, we get $x^4 + 2 + 1/x^4 = 16$. But $(x^2 - 1/x^2)^2 = x^4 - 2 + 1/x^4 = 12$. It follows that $x^2 - 1/x^2 = \pm\sqrt{12}$.

Alternate solution: Multiply both sides of $x^2 + 1/x^2 = 4$ by x^2 and rearrange to obtain $x^4 - 4x^2 + 1 = 0$. Use the quadratic formula to solve for x^2: $x^2 = 2 \pm \sqrt{3}$. Substitute into $x^2 - 1/x^2$ and simplify to obtain $\pm 2\sqrt{3}$.