READER reflections

2010: A MAGIC YEAR?

Can you and your students create a table whose main diagonals will always yield the sum of 2010? Follow these steps and then challenge yourself and your students to determine why the sum is always 2010. In keeping with the spirit of NCTM's Standards, have your students communicate their work verbally as well as in writing. Have fun!

Begin by numbering the squares of a 12×12 chessboard from 1 through 144 (see **table 1 [Kuenzi-Sriskandarajah]**).

Next, arbitrarily interchange any number of rows of the chessboard to obtain a new "scrambled" numbering of the chessboard squares (see **table 2 [Kuenzi-Sriskandarajah]**).

Then arbitrarily interchange columns of the modified chessboard in any fashion to obtain a further "scrambled" numbering (see **table 3 [Kuenzi-Sriskandarajah]**).

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged, but because of the large number submitted, we do not send letters of acceptance or rejection. Letters to be considered for publication should be in MS Word document format and sent to **mt@nctm.org**. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including zip or postal code and e-mail address, per the style of the section.

Now add 95 to every number on the chessboard (see **table 4 [Kuenzi-Sriskandarajah]**).

Add the twelve numbers on either main diagonal to get the amazing sum of 2010. Following are two examples:

$$155 + 232 + 214 + 188 + 97 + 199 + 225 + 137 + 159 + 126 + 170 + 108 = 2010$$

and

$$144 + 230 + 210 + 183 + 101 + 201 + 223 + 133 + 164 + 130 + 172 + 119 = 2010$$

For an even more amazing result, highlight any number on the chessboard and cross out all the other numbers in the same row and column as the highlighted number (see **table 5 [Kuenzi-Sriskandarajah]**).

Repeat this operation until twelve numbers remain (see **table 6 [Kuenzi-Sriskandarajah**].

Add the twelve highlighted numbers to obtain 2010:

$$239 + 160 + 178 + 140 + 181 + 127 + 153 + 113 + 207 + 102 + 218 + 192 = 2010$$

Different students and groups will have different tables and different addends, but the sum will always be 2010.

Why is the sum always 2010?

You and your students may come up with some interesting ideas as to why the sum is always 2010. Of course, their explanations will depend on their mathematical background, but it is important to give them an opportunity to understand the mathematics behind these results. What follows is one possible explanation.

In the original table (**table 1 [Kuenzi-Sriskandarajah]**), every entry in the second row is 12 more than the corresponding entry in the first row; every entry in the third row is 24 more than the corresponding entry in the first row; and so on for the other rows. For the original table, then, the number in column c and row r is given by the formula $c + 12(r - 1)$. After we interchange rows and then interchange columns, the numbers are still of this form, but c and r refer to the entry's original column and row numbers. If we add 95 to every entry in the table, the numbers are of the form $95 + c + 12(r - 1)$. After obtaining the twelve highlighted numbers, we note that exactly one number is left in each row and exactly one number is left in each column. So the sum is as follows:

$$(12) \cdot (95) + (1 + 2 + \cdots + 12) + 12(0 + 1 + 2 + \cdots + 11) = 1140 + 78 + (12)(66) = 2010$$

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Table 1 (Kuenzi-Sriskandarajah)

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99	100	101	102	103	104	105	106	107	108
109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132
133	134	135	136	137	138	139	140	141	142	143	144

Table 2 (Kuenzi-Sriskandarajah)

49	50	51	52	53	54	55	56	57	58	59	60
133	134	135	136	137	138	139	140	141	142	143	144
109	110	111	112	113	114	115	116	117	118	119	120
85	86	87	88	89	90	91	92	93	94	95	96
1	2	3	4	5	6	7	8	9	10	11	12
97	98	99	100	101	102	103	104	105	106	107	108
121	122	123	124	125	126	127	128	129	130	131	132
37	38	39	40	41	42	43	44	45	46	47	48
61	62	63	64	65	66	67	68	69	70	71	72
25	26	27	28	29	30	31	32	33	34	35	36
73	74	75	76	77	78	79	80	81	82	83	84
13	14	15	16	17	18	19	20	21	22	23	24

Table 3 (Kuenzi-Sriskandarajah)

60	53	59	57	50	56	58	54	52	55	51	49
144	137	143	141	134	140	142	138	136	139	135	133
120	113	119	117	110	116	118	114	112	115	111	109
96	89	95	93	86	92	94	90	88	91	87	85
12	5	11	9	2	8	10	6	4	7	3	1
108	101	107	105	98	104	106	102	100	103	99	97
132	125	131	129	122	128	130	126	124	127	123	121
48	41	47	45	38	44	46	42	40	43	39	37
72	65	71	69	62	68	70	66	64	67	63	61
36	29	35	33	26	32	34	30	28	31	27	25
84	77	83	81	74	80	82	78	76	79	75	73
24	17	23	21	14	20	22	18	16	19	15	13

Table 4 (Kuenzi- Sriskandarajah)

155	148	154	152	145	151	153	149	147	150	146	144
239	232	238	236	229	235	237	233	231	234	230	228
215	208	214	212	205	211	213	209	207	210	206	204
191	184	190	188	181	187	189	185	183	186	182	180
107	100	106	104	97	103	105	101	99	102	98	96
203	196	202	200	193	199	201	197	195	198	194	192
227	220	226	224	217	223	225	221	219	222	218	216
143	136	142	140	133	139	141	137	135	138	134	132
167	160	166	164	157	163	165	161	159	162	158	156
131	124	130	128	121	127	129	125	123	126	122	120
179	172	178	176	169	175	177	173	171	174	170	168
119	112	118	116	109	115	117	113	111	114	110	108

Table 5 (Kuenzi-Sriskandarajah)

155	148	154	152	145		153	149	147	150	146	144
239	232	238	236	229		237	233	231	234	232	228
215	208	214	212	205		213	209	207	210	206	204
191	184	190	188	181		189	185	183	186	182	180
107	100	106	104	97		105	101	99	102	98	96
					199						
227	220	226	224	217		225	221	219	222	218	216
143	136	142	140	133		141	137	135	138	134	132
167	160	166	164	157		165	161	158	162	158	156
131	124	130	128	121		129	125	123	136	122	120
179	172	178	176	169		177	173	171	174	170	168
119	112	118	116	109		117	113	111	114	110	108

Table 6 (Kuenzi-Sriskandarajah)

						153					
239											
								207			
				181							
									102		
											192
										218	
			140								
	160										
					127						
		178									
							113				

AN INTRIGUING EXPONENTIAL INEQUALITY: ANOTHER SOLUTION

I loved the problem presented in John Robert Perrin's "An Intriguing Exponential Inequality" (*MT* August 2009, vol. 103, no. 1, pp. 50–55) and write to suggest an alternative analytic solution.

My solution proceeds immediately from the realization during graphical exploration that the case of interest is the case where the graph of $f(x) = a^x$ has the line $L(x) = x$ as a tangent at some point where $x = c$. Since the slope of this tangent line is 1 and since $f'(x) = a^x \ln a$, we know that

$$a^c \ln a = 1. \qquad (1)$$

If we think about the point of tangency as being a point on f, we think of its coordinates as (c, a^c). If, however, we think of the point of tangency as being a point on L, we think of its coordinates as (c, c). Since these are two different ways of naming the same point, we can equate the y-coordinates:

$$a^c = c \qquad (2)$$

Substituting (2) into (1), we have

$$c \ln a = 1. \qquad (3)$$

Taking the log of both sides of (2) gives

$$c \ln a = \ln c. \qquad (4)$$

Substituting (4) into (3) gives

$$\ln c = 1,$$

which means that $c = e$. Putting this result into (2) yields

$$a^e = e.$$

Consequently, we find the same key value of a found in the article: $a = e^{1/e}$.

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A FOURTH WAY TO BREAK A STICK
A Combinatoric Method

In my article "Three Ways to Break a Stick" (*MT* August 2009, vol. 103, no. 1, pp. 56–61), I showed that the probability that the pieces of a stick randomly broken in two places can form a triangle is the same as the probability that three random points on a circle do not fall in a semicircle.

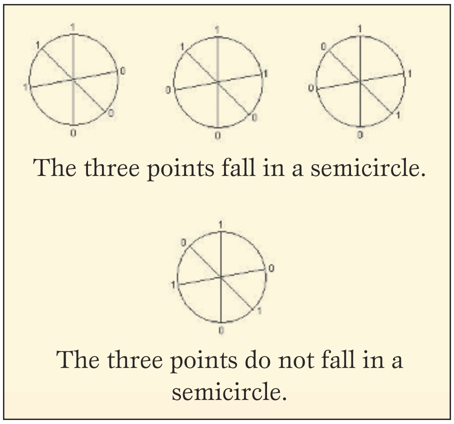

Fig. 1 (Bannon)

Not mentioned in the article is a combinatoric method of finding the probability that three random points on a circle fall in a semicircle. Label each of the random points as 1. From each point, draw a diameter and label the other endpoint of the diameter as 0. If we orient the circle in such a way that a 1 is in the twelve o'clock position and if we travel around the circle in a clockwise direction, only four patterns of 1s and 0s are possible (see **fig. 1 [Bannon]**). Three of these four patterns have three consecutive 1s, indicating that the points fall in a semicircle. We can conclude that the probability that three random points on a circle fall in a semicircle is 3/4 and that the probability that they do not is 1/4.

Thus, we have a fourth way of showing that the probability that the pieces of a stick randomly broken in two places can form a triangle is 1/4.

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Conditional Probability

Bannon's article "Three Ways to Break a Stick" offers three solutions for finding the probability that the pieces of a stick will form a triangle, and all three use an approach that considers the intersection of both breaks. Another approach makes use of conditional probability—that is, looking at the probability of the second break, Y, given the first break, X.

If we use the interval (0, 1) to represent that the stick has a uniform probability distribution for each break—that is, that every point in the interval is equally likely—the probability density function is

$$f(x) = \frac{1}{b-a} = \frac{1}{1-0}.$$

So the probability that a break lands in some interval $[c, d] \in (0, 1)$ is simply the distance $d - c$:

$$P(c \le X \le d) = \int_c^d \frac{1}{1-0}\,dx = x\Big|_c^d = d - c$$

If we use conditional probability, there are two disjoint cases for Y that would fail to make a triangle:

1. Y lands on the same half as X. Without loss of generality, assume that X is on the left half. The probability that Y lands on the same half, in the interval (0, 0.5), is $0.5 - 0 = .5$.
2. Y lands on the opposite half as X and is farther than 0.5 away from X. Again, if we assume that X is on the left half, then the probability that Y is farther than $x + 0.5$ (red) is the same as the distance from x to the middle: $|x - 0.5|$ (see **fig. 1 [(Wasserman]**).

The probability that the pieces do not form a triangle is the sum of these disjoint cases:

$$P\left(Y \text{ causes no triangle} \middle| X\right) = 0.5 + \left|x - 0.5\right|$$

Since $X \in (0, 1)$ and is a continuous random variable, the sum of all the individual probabilities that Y causes no triangle is the integration

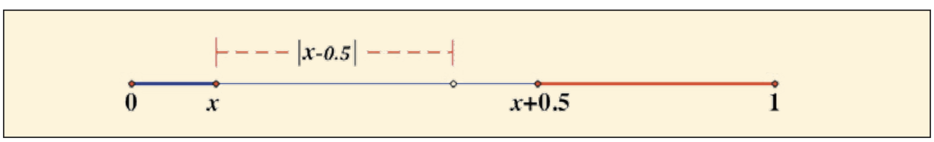

Fig. 1 (Wasserman)

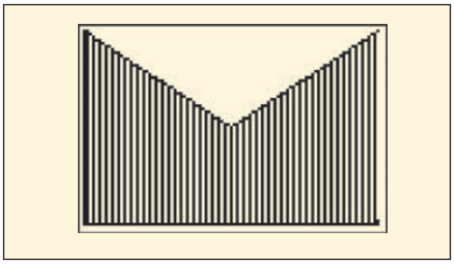

Fig. 2 (Wasserman)

$$\int_0^1 \left(0.5+\left|x-0.5\right|\right)dx = \int_0^1 0.5\,dx + \int_0^{0.5} -x+0.5\,dx$$
$$+\int_{0.5}^1 x-0.5\,dx$$
$$=0.75,$$

making the probability of forming a triangle .25 (see **fig. 2 [Wasserman]**).

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9/10^n – 1

In his excellent article "Soft Drinks, Mind Reading, and Number Theory" (*MT* November 2009, vol. 103, no. 4, pp. 278–83), Kyle T. Schultz discusses a student's assertion that "one could express each power of 10 as a sum of 1 and a multiple of 9." Schultz writes: "This reasoning seemed plausible but left a new conjecture for us to wrestle with: A power of 10 can always be expressed as a sum of 1 and a multiple of 9. This conjecture could be explored by investigating factoring the expression $10^n - 1$ by using differences of squares and cubes" (p. 283).

To me, a more natural way would be to expand 10^n as $(9+1)^n$ using the binomial theorem:

$$(9+1)^n = 1\cdot 9^n \cdot 1^0 + n\ \cdot 9^{n-1}\cdot 1^1 + \binom{n}{2}\cdot 9^{n-2}\cdot 1^2+\cdots+1\cdot 9^0\cdot 1^n$$
$$= 9^n + n\ \cdot 9^{n-1} + \binom{n}{2}9^{n-2}+\cdots+9n+1$$

It then becomes clear that each term except the last is divisible by 9, so that when 1 is subtracted, the result is divisible by 9.

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VOLUME OF THE FRUSTUM GENERALIZED

This note generalizes the work of both Javad H. Zadeh in his reflection "Egyptian Geometry" (*MT* September 2008, vol. 102, no. 2, pp. 86–87) and Mark Snyder in his reflection "Frustum of a Pyramid Revisited" (*MT* August 2009, vol. 103, no.1, pp. 7–8), who gives alternative solutions for $n = 3$. Snyder also gives an alternative proof for the volume of a right square pyramid.

We first introduce some terminology. An n-cube is an n-dimensional cube. The usual cube is a 3-cube. The square is a 2-cube. A line segment is a 1-cube. An n-right square pyramid (n-pyramid for short) is an n-dimensional figure formed with an $(n - 1)$-cube as base and line segments joining a point not on the base to the vertices of the base such that the line segment from the point to the center of the base is perpendicular to the base. The usual right square pyramid is a 3-pyramid. The isosceles triangle is a 2-pyramid. The face of an n-cube is one of the $(n - 1)$-cubes that form its boundary. The frustum of an n-pyramid is the figure formed by cutting an n-pyramid with an $(n - 1)$-dimensional flat that is parallel to the base and discarding the smaller n-pyramid that is formed. The frustum of a 2-pyramid is an isosceles trapezoid. The frustum of a 3-pyramid is the usual one.

Note that in 2-space, if we join the center of a 2-cube (square) to the vertices, we cut the square into four congruent 2-pyramids (isosceles triangles) whose bases are the faces of the 2-cube (square). In 3-space, if we join the center of the cube to the vertices of the cube, we cut the cube into six congruent pyramids whose bases are the faces of the cube. In general, an n-cube has $2n$ faces that are $(n - 1)$-cubes, so when we join the center of the n-cube to the vertices, we cut the n-cube into $2n$ congruent n-pyramids.

Now consider the n-cube in which each side has length B. The volume of the n-cube is B^n. The n-pyramid has an $(n - 1)$-cube as its base and height H. Following Snyder, from dimensional considerations we find that the volume of the n-pyramid must be $V = kB^{n-1}H$, where k is a constant. Since this result holds for all n-pyramids, it holds for n-pyramids of height $B/2$, which are simply the pyramids we get when we split the n-cube into n-pyramids by joining the center of the n-cube to each of its vertices. So for such n-pyramids,

$$V = kB^{n-1}\left(\frac{B}{2}\right)=\frac{kB^n}{2}.$$

Since there are $2n$ faces and there is one pyramid for each face, the volume of the n-cube is $B^n = (2n)\cdot(kB^n/2)$. So $k = 1/n$. Therefore, the volume of the n-pyramid is given by $V = B^{n-1}H/n$.

To determine the volume of the frustum of the n-pyramid, we follow Zadeh's derivation for $n = 3$. Let the height of the frustum be h and the height of the n-pyramid be $H = h + k$. Then the volume of the frustum is the volume of the n-pyramid minus the volume of the small n-pyramid that is cut off to form the frustum:

$$\begin{aligned}V &= \frac{1}{n}(h+k)B^{n-1}-\frac{1}{n}(kb^{n-1})\\ &= \frac{1}{n}[(h+k)B^{n-1}-kb^{n-1}]\\ &= \frac{1}{n}(hB^{n-1}+kB^{n-1}-kb^{n-1)})\\ &= \frac{1}{n}[hB^{n-1}+k(B^{n-1}-b^{n-1})]\\ &= \frac{1}{n}[hB^{n-1}+k(B-b)\,\cdot\\ &\quad (B^{n-2}+B^{n-3}b+B^{n-4}b^2+\ldots+b^{n-2})]\end{aligned}$$

Now, as Zadeh did, we use similar triangles to get $k = hb/(B - b)$. Substituting in the formula above, we get the following:

$$V=\frac{1}{n}\left[\begin{array}{l}hB^{n-1}+\dfrac{hb}{B-b}(B-b)(B^{n-2}+\\ \quad B^{n-3}b+B^{n-4}b^2+\ldots+b^{n-2})\end{array}\right]$$

Factoring out h and simplifying, we get the desired formula:

$$V=\frac{1}{n}h(B^{n-1}+B^{n-2}b+B^{n-3}b^2+\ldots+b^{n-1})$$

Snyder's derivation of the formula for the volume of the frustum works for $n = 2$, 3, and 4, but its extension to higher dimensions does not seem possible: Too many unknowns come into play, and too few symmetries are given to allow solving for all these unknowns. Too bad, because it is a lovely argument.

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OBTAINING THE SUM WITHOUT USING INDUCTION

In their reflection "Obtaining the Sum without Using Induction" (*MT* September 2009, vol. 103, no. 2, pp. 96, 98), Richard Grassl and Thomas Koshy present a method for decomposing certain sums without using induction. I would like to introduce a different method that also does not depend on induction.

Assume that S is the sum of a finite power series

$$\sum_{i=1}^{n} ix^{i-1}.$$

If $x = 1$, then

$$S = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

If $x \neq 1$, then

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} \quad (1)$$

and

$$xS = x \ + 2x^2 + 3x^3 + 4x^4 + \cdots + (n-1)x^{n-1} + nx^n. \quad (2)$$

Shifting xS one column to the right and subtracting (2) from (1) produces

$$(1-x)S = 1 + x + x^2 + x^3 + \cdots + x^{n-1} - nx^n.$$

The first n terms of the result form a geometric series with common ratio x. Therefore,

$$(1-x)S = 1 + x + x^2 + x^3 + \cdots + x^{n-1} - nx^n.$$

$$\begin{aligned}(1-x)S &= \frac{1-x^n}{1-x} - nx^n \\ &= \frac{1-x^n-nx^n+nx^{n+1}}{1-x} \\ &= \frac{[n(x-1)-1]x^n+1}{1-x}.\end{aligned}$$

Dividing both sides of the equation by $(1 - x)$, we obtain

$$S = \sum_{i=1}^{n} ix^{i-1} = \frac{[n(x-1)-1]x^n+1}{(x-1)^2}$$

and

$$\sum_{i=1}^{n} ix^{i} = x\sum_{i=1}^{n} ix^{i-1} = \frac{[n(x-1)-1]x^{n+1}+x}{(x-1)^2}.$$

This method can be applied to find the sum of a finite power series whose coefficients have low-order finite differences.

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IT'S NEVER TOO EARLY

Russell Gordon of Ontario, Canada, used his iPhone to snap this photograph of his one-year-old daughter, Claire Dorothy Gordon, intently perusing the pages of his *MT*. Claire must have enjoyed "reading" the journal, because shortly afterward she began to eat it. We appreciate readers who digest the journal from cover to cover, although not, perhaps, exactly in this way.—*Ed.*

It's never too early to begin digesting mathematical concepts.