APRIL

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The diameter of the larger circle is 10 units, and the diameter of the smaller circle is 1/5 that of the larger circle. The two congruent rectangles, whose corresponding sides are perpendicular, have length 8 units and width 2 units. Find the exact area of the shaded region.

1

Farmer Abe wants to fence in a rectangular area on his farm next to a river. He has 2000 feet of fencing and does not need fencing along the river. He further wants to subdivide the field into two smaller rectangular fields of equal area. How should he configure the fencing to maximize the enclosed area?

2

In the following expression, each letter represents a different integer from 0 through 9, except that 5 is not included. If S = 2 and M = 7 and the value of IS is less than 50, find the integer that each letter represents.

```
  M A T H
+     I S
---------
  C O O L
```

3

A dartboard has three concentric circles with radii 4 in., 8 in., and 14 in. The smallest circle is red, the ring surrounding that circle is white, and the outer ring is blue. A person with no skill throws a dart at the board. If the dart hits the board, what is the probability that it hits the red or blue regions?

4

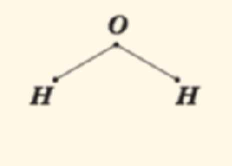
Rotations and reflections of a plane figure are *symmetries* if the result of the transformation cannot be detected by a viewer. For example, a reflection in an altitude of an equilateral triangle leaves the triangle apparently unchanged, as do rotations of 120°, 240°, or 360° about the center. Thus, an equilateral triangle has 6 symmetries. How many symmetries does a regular hexagon have?

5

Dan has pennies, nickels, dimes, and quarters in his pocket—a total of 36 coins. He has twice as many nickels as dimes and four times as many pennies as quarters. If he has \$2.27 in coins, how many coins of each type does he have?

6

Describe the symmetries of a water molecule as indicated in the figure.



7

In the following addition problem, each letter represents a different integer from 0 through 9, inclusive. If G = 0 and N = 4, find the rest of the values to crack the code for the following:

```
      T E N
    N I N E
  E I G H T
-----------
  S E V E N
```

8

Lisa checks the statistics for her favorite puzzle game on her iPod. She sees that she has won exactly 72% of the 2400 puzzles that she has played. What is the minimum number of puzzles that Lisa must win before she has won at least 73% of the total number of puzzles played?

9

Signs in a department store read, "75% off original prices when you take an extra 50% off already reduced prices." By what percent were original prices reduced before today's sale?

10

Consider the following linear equation:

$$4x + 11y = 2011$$

How many solutions are there such that x and y are positive integers? (Note the date: 4/11/2011.)

11

Alex and Barrie run in a marathon. Alex runs at a steady pace of 7 mph, while Barrie uses the Galloway method of racing, running 4 minutes at 8 mph and then walking for 1 minute at 3.5 mph. If each maintains this pace for the entire race, who will finish the race first?

12

The centers of two congruent tangent circles (shown in blue) lie on a diameter of a larger circle. The centers of six smaller congruent tangent circles lie on the same diameter. What fraction of the area of the largest circle is shaded blue?

13

Let x and y be real numbers. Find the area enclosed by the graph of the following equation:

$$|2x| + |2y| = 1$$

14

A store sold 60 DVDs. Some of these were on sale and sold for \$4.95 each, while the rest were priced at \$5.95 each. If the receipts for DVDs at the end of the day totaled \$317.00, how many of each type were sold?

15

Express the decimal $0.\overline{2541}$ as a ratio of two integers.

16

A scientist is testing a new form of pathogen designed to eliminate a certain type of bacteria. At the start of the experiment, the population of the bacteria colony was 10,000. After 3 hours, 2000 bacteria were killed. If the die-off can be modeled by a continuous exponential function, how long will it take for the bacteria population to fall to 5000?

17

The area enclosed by the graph of

$$|ax| + |ay| = 1$$

is 18 units². Given $a > 0$, find the value of a.

18

Barstucks is making a new coffee blend. The company bought 50 lbs. of Kona beans at \$34.95 per lb. and plans to mix these beans with Colombian beans that sell for \$7.99 per lb. wholesale. If Barstucks wants its cost of the blend to be \$11.95 per lb., how many lbs. of Colombian beans should it buy?

19

A license plate manufacturer is punching plates with the pattern

L#L-##LL

where L represents a letter and # represents a digit from 1 through 9. If a letter can be any letter from A to Z except O, how many different license plates are possible?

20

A radio tower sends a signal with a 12-mile radius. Will a house that is 10 miles east and 5 miles north of the tower receive the signal?

21

Violet, Kim, Lucy, Charles, and John are at a barbeque with their favorite foods: hot dogs, hamburgers, meatless pasta salad, veggie burgers, and ribs. Charles and Kim are carnivores, but neither likes hamburgers. Violet is a vegetarian. The women dislike ribs. Lucy loves pasta salad. If no two people have the same favorite food, match the people and the food.

22

Using rules of algebra, simplify the following expression:

$$\left(\frac{a^3x^2y^{-3}}{b^2x^{-3}y^2}\right)^{-1} \cdot \left(\frac{ax^2}{by^3}\right)^2 \div \frac{1}{(axy)^2}$$

23

The leftmost digit of a five-digit number is twice the next digit, and the third digit is 1 more than the sum of the two digits to its left. The fourth digit is different from the digits to its left but the same as the digit to its right. If the sum of the five digits is 15, what is the number?

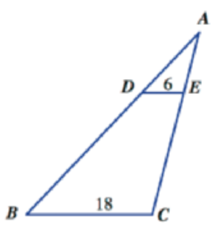
24

Andrei Nikolaevich Kolmogorov, born on this date in 1903, is considered by some to be the greatest Russian mathematician of the twentieth century. Kolmogorov devised today's problem at the age of five:

How many different two-dimensional patterns can you create with thread while sewing on a four-hole button?

25

In $\triangle ABC$, $\overline{DE} \parallel \overline{BC}$. $DE = 6$, $BC = 18$, and $AD = 12$. If the perimeter of $\triangle ABC = 81$, what is the length of $\overline{EC}$?



26

During a fireworks show, a sunburst rocket is launched vertically into the air at 90 m/sec. If the rocket explodes when its velocity is zero, how long will it be in the air until it explodes? Assume that

acceleration $a = \dfrac{-9.8 \text{ m/sec.}}{\text{sec.}}$.

27

Find all real solutions to the following equation:

$$\left(x^2 - 7x + 11\right)^{x^2 - 2x - 35} = 1$$

28

Without using a calculator, find the distance between the two points with polar coordinates

$$(r_1, \theta_1) = \left(6, \frac{22\pi}{7}\right) \text{ and } (r_2, \theta_2) = \left(-8, \frac{37\pi}{14}\right).$$

29

A bowl contains 8 marbles: 3 red, 2 blue, 1 white, and 2 green. If 3 marbles are randomly selected without replacement, what is the probability that no green marble is selected?

30

SOLUTIONS to calendar

Looking for more "Calendar" problems?
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The Editorial Panel of the *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly "Calendar." Send problems to the "Calendar" editors. Remember to include a complete solution for each problem submitted.

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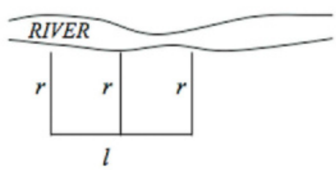
1. $24\pi - 28$ units^2. The area of the bigger circle with radius 5 is $\pi \cdot 5^2 = 25\pi$ units^2, and the area of the smaller circle is $\pi \cdot 1^2 = \pi$ units^2. The area of each rectangle is $8 \cdot 2 = 16$ units^2. Their overlap forms a 2×2 square that should not be counted twice: $2 \cdot 16 - 4 = 28$ units^2. The area of the shaded region is the area of the bigger circle minus the areas of the other objects: $25\pi - \pi - 28 = 24\pi - 28$ units^2.

2. 1000/3 ft. $\times$ 1000 ft. with a 1000/3 ft. fence down the center perpendicular to the river. We consider the following two cases: (1) the fence in the interior is perpendicular to the river; or (2) the fence in the interior is parallel to the river.

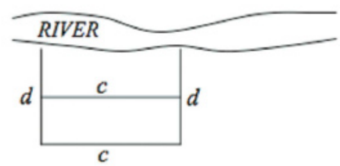
Case 1: Let r = the fence length perpendicular to the river, and let l = the fence length parallel to the river. Then $3r + l = 2000 \rightarrow l = -3r + 2000$. The enclosed area $= rl$. We rewrite the latter equation: area $= r(-3r + 2000) \rightarrow$ area $= -3r^2 + 2000r$. The graph of this quadratic equation is a parabola that opens downward. The maximum value occurs at the vertex when

$$r = \frac{-2000}{2(-3)} = \frac{1000}{3} \text{ and } l = 1000.$$

Then the maximum area $= rl = 1000^2/3$ ft.^{2}.



Case 2: Let d = the fence length perpendicular to the river, and let c = the fence length parallel to the river. Then $2d + 2c = 2000 \rightarrow d = -c + 1000$. The enclosed area $= cd$. We rewrite the latter equation: area $= c(-c + 1000) \rightarrow$ area $= -c^2 + 1000c$. The maximum value occurs when $c = -1000/-2 = 500$ and when $d = 500$. The maximum area is then $cd = 500^2 = 250{,}000$ ft.^{2}.



The first configuration allows a larger area to be enclosed. The fence encloses a field 1000/3 ft. $\times$ 1000 ft. with a 1000/3 ft. length of fencing down the center.

3. A = 9, T = 6, H = 1, I = 4, C = 8, O = 0, and L = 3. Replace M and S with 7 and 2, respectively:

$$\begin{array}{r} 7\ A\ T\ H \\ +\quad I\ 2 \\ \hline C\ O\ O\ L \end{array}$$

Since the thousands place in the sum is C and not M, we know that 1 was carried from the hundreds place. Therefore, C = 8. We use the same reasoning to conclude that O = A + 1 and that A + 1 > 9 implies A > 8. Therefore, A = 9 and A + 1 = 10, which implies that O = 0. We now have the following:

$$\begin{array}{r} 7\ 9\ T\ H \\ +\quad I\ 2 \\ \hline 8\ 0\ 0\ L \end{array}$$

The 0 in the tens place in the sum cannot be the result of 0 + 0; we conclude that either T + I = 10 or T + I + 1 = 10. The digits 1, 3, 4, and 6 remain to be assigned to the letters T, H, I, and L. If T + I = 10, then T = 6 and I = 4. (Recall that IS < 50.) That leaves H = 1 and L = 3. If we try to assign H = 4 and L = 6, we obtain T + I = 4. We reject this assignment because we have 0 in the tens place in the sum.

4. 37/49. To answer this geometric probability question, we find the ratio of the sum of the red and blue areas to the area of the entire dartboard. The area of the red circle is 16π. The area of the white ring is $64\pi - 16\pi = 48\pi$. The area of the blue ring is $196\pi - 64\pi = 132\pi$. Therefore, we have the following:

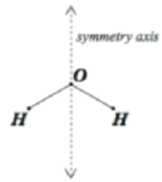
$$\begin{aligned}\frac{\text{area red circle} + \text{area blue ring}}{\text{area entire dartboard}} &= \frac{16\pi + 132\pi}{196\pi}\\ &= \frac{148\pi}{196\pi} = \frac{37}{49}\\ &\approx .7551\end{aligned}$$

Alternate solution: Subtract the area of the white ring from the total area and divide by the total area.

5. 12 symmetries. A regular hexagon has 12 symmetries: 6 rotations (60°, 120°, 180°, 240°, 300°, and 360°) about the center, 3 reflections in the perpendicular bisectors of the sides, and 3 reflections in lines through opposite vertices.

6. 12 pennies, 14 nickels, 7 dimes, and 3 quarters. If d = the number of dimes, then $2d$ = the number of nickels. If q = the number of quarters, then $4q$ = the number of pennies. Dan has 36 coins: $d + 2d + q + 4q = 36 \rightarrow 3d + 5q = 36$. Since d and q must be positive integers, the equation has 2 solutions: $d = 2$, $q = 6$ and $d = 7$, $q = 3$. If Dan has 2 dimes and 6 quarters, then he also has 4 nickels and 24 pennies, for a total of \$2.14. If he has 7 dimes and 3 quarters, then he also has 14 nickels and 12 pennies, for a total of \$2.27. The latter solution satisfies all the stated conditions.

7. Identity, reflection in symmetry axis, and rotations of 180° and 360° about the symmetry axis. Finding the symmetry axis for the molecule facilitates describing its symmetries. The symmetry axis is the bisector of $\angle HOH$ (in the plane of the molecule). Reflection in the symmetry axis is a symmetry of the molecule. Rotations of 180° and 360° about the axis are symmetries. The identity transformation leaves the water molecule unchanged. Thus, a water molecule has a total of 4 symmetries.



8. T = 8, E = 2, I = 7, H = 5, S = 3, and V = 6. We substitute 0 for G and 4 for N to obtain the following:

$$\begin{array}{ccccc} & & T & E & 4\\ & 4 & I & 4 & E\\ E & I & 0 & H & T\\ \hline S & E & V & E & 4\end{array}$$

The digits in the ones column imply that E + T = 10. We will carry a 1 to the tens column, which gives us another equation: 1 + E + 4 + H = E + 10. We can solve the second equation for H, finding H = 5. The digits in the ten-thousands place are different; therefore, a 1 must be carried *from* the thousands place. Since we do not know whether a 1 was carried *to* the thousands place, we consider two possibilities: 4 + I = E + 10 or 1 + 4 + I = E + 10. If the former is true, I = E + 6, which implies that E ≤ 3. If the latter is true, I = E + 5, which implies that E ≤ 4. But N = 4, so E must be 1, 2, or 3. If E = 3, then S = 4, but we have N = 4. We test E = 2, which implies that T = 8:

$$\begin{array}{ccccc} & & T & 2 & 4\\ & 4 & I & 4 & 2\\ 2 & I & 0 & 5 & T\\ \hline S & 2 & V & 2 & 4\end{array} \rightarrow \begin{array}{ccccc} & & 8 & 2 & 4\\ & 4 & I & 4 & 2\\ 2 & I & 0 & 5 & 8\\ \hline 3 & 2 & V & 2 & 4\end{array}$$

Now we see that a 1 was carried to the thousands place. We have 1 + 4 + I = 12, so I = 7. Finally, V = 6. We leave it to the reader to find the contradiction in the case not shown—namely, E = 1.

9. 89. Achieving a winning percentage of at least 73% after playing as few additional puzzle games as possible implies that Lisa wins every additional game she plays. She has already won $0.72 \cdot 2400 = 1728$ games. Let w = additional number of games won. We have the following:

$$\begin{aligned}&\frac{1728 + w}{2400 + w} \geq 0.73 \rightarrow\\ &1728 + w \geq 0.73(2400 + w) \rightarrow\\ &1728 + w \geq 1752 + 0.73w \rightarrow\\ &0.27w \geq 24 \rightarrow w \geq 88\frac{8}{9}\end{aligned}$$

Lisa must win at least 89 consecutive additional puzzle games.

10. 50%. If p = the original price, then $0.25p$ = the final sale price. If r = the first reduction or discount expressed as a decimal, then $(1 - r)p$ = the price marked after first reduction. We reduce this sale price by 50%: $0.5(1 - r)p = 0.25p$. We see, of course, that the original price has no bearing on the problem. We have the following: $0.5(1 - r) = 0.25 \rightarrow 0.5 - 0.5r = 0.25 \rightarrow r = 0.5 = 50\%$. We confirm this answer: Two successive 50% discounts are equivalent to one 75% discount.

11. 46 solutions. If we let $y = 1$, then $4x = 2000$ and $x = 500$. This solution—

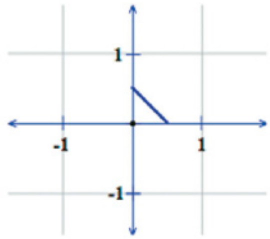
(500, 1)—has the smallest y-value in the set of positive integral solutions. We find the other solutions by considering the slope of the line $4x + 11y = 2011$. The slope $m = -4/11$ implies that the solution with the next smallest value of y is $(500 - 11, 1 + 4)$, or (489, 5). Additional solutions include $(500 - 2 \cdot 11, 9)$, $(500 - 3 \cdot 11, 13)$, $(500 - 4 \cdot 11, 17)$, ..., $(500 - k \cdot 11, 1 + k \cdot 4)$ where k is a natural number and $500 - k(11) > 0$. We solve this inequality: $k < 500/11 \rightarrow k < 45\ 5/11$. Setting $k = 45$, we find the solution with the smallest x-value in the set of positive integral solutions: $(500 - 45 \cdot 11, 1 + 45 \cdot 4)$, or (5, 181). Thus, the number of positive integral solutions is $45 + 1 = 46$.

Alternate solution: Use modular arithmetic. Start by setting $4x \equiv 2011 \pmod{11}$, which reduces to $x \equiv 5 \pmod{11} \rightarrow x = 11t + 5$, where t is an integer. Then substitute that for x in the original equation to obtain $y = 181 - 4t$. Values of t from 0 to 45 inclusive will then produce positive integral values for x and y.

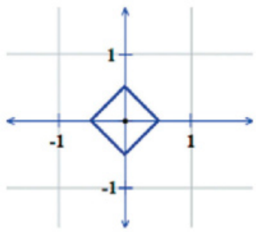
12. Barrie. Alex runs the entire race at a constant rate of 7 mph while Barrie runs 4/5 of the race at 8 mph and 1/5 at 3.5 mph. Barrie's average rate is $(4/5) \cdot 8 + (1/5) \cdot 3.5 = 7.1$ mph, which is faster than Alex's rate.

13. One-third of the figure is shaded blue. Let the radius of the smallest circles be $r_1 = 1$; the area of each of these circles is π. The radius of each blue circle is $r_2 = 3$, and the area of each blue circle is 9π. The radius of the largest circle is $r_3 = 6$, and its area is 36π. The area shaded blue is twice the area of a blue circle minus six times the area of the smallest circle, or $2 \cdot 9\pi - 6\pi = 12\pi$. Therefore, the required ratio is $12\pi/36\pi = 1/3$.

14. One-half. We will make use of the algebraic definition of absolute value to examine the behavior of the graph in each quadrant. By definition, $|x| = x$ for $x \geq 0$, and $|x| = -x$ for $x < 0$. We rewrite the given equation: $|2x| + |2y| = 1 \rightarrow 2|y| = -2|x| + 1 \rightarrow |y| = -|x| + 1/2$. For $x, y \geq 0$, this equation is equivalent to $y = -x + 1/2$, a line segment in quadrant I with slope -1 and y-intercept 1/2.



For $y > 0$ and $x < 0$, the equation is equivalent to $y = x + 1/2$, a line segment in quadrant II with slope 1 and y-intercept 1/2. For $y < 0$ and $x > 0$, the equation is equivalent to $-y = -x + 1/2 \rightarrow y = x - 1/2$. This graph will appear in quadrant IV as a segment with slope 1 and y-intercept $-1/2$. Finally, for $x, y < 0$, the equation is equivalent to $-y = -(-x) + 1/2 \rightarrow y = -x - 1/2$, a line segment in quadrant III with slope -1 and y-intercept $-1/2$.



The slopes of consecutive segments are negative reciprocals, thus confirming what we see: The enclosed region is a square. The intercepts are 1/2 and $-1/2$, so the diagonals have length 1. The area of the square is half the product of the diagonals—in this case, $1/2 \cdot 1 \cdot 1 = 1/2$.

15. 20 DVDs at \$5.95 each and 40 DVDs at \$4.95 each. Let s represent the number of DVDs sold at the sale price, and let r represent the number sold at the regular price. We have a system of two equations:

$$5.95r + 4.95s = 317.00$$
$$r + s = 60$$

Substituting $60 - r$ for s in the first equation produces $5.95r + 4.95(60 - r) = 317.00$, which yields values of 20 and 40 for r and s, respectively.

16. 77/303. Let $x = 0.25412541....$ The repetend has 4 digits. We multiply both sides of this equation by 10^4, so the digits after the decimal points remain aligned. We have the following:

$$\begin{array}{rcl} 10{,}000\ x & = & 2541\ .2541... \\ x & = & 0\ .2541... \\ \hline 9999\ x & = & 2541 \end{array}$$

So $x = 2541/9999 = 847/3333 = 77/303$.

17. Around 9.32 hours. Consider the general form of the exponential growth/decay function: $P = P_0e^{kt}$ where P_0 is the original population, k is the growth rate, and t is time (here, in hours). If we are given the original population of 10,000 bacteria and a population loss of 2000 after 3 hours, then we can solve for k, using $P = 8000$, $P_0 = 10{,}000$, and $t = 3$:

$$\begin{aligned} 8000 &= 10{,}000e^{3k} \\ 0.8 &= e^{3k} \\ \ln(0.8) &= 3k\ln(e) \\ \ln(0.8) &= 3k \\ \frac{\ln(0.8)}{3} &= k \end{aligned}$$

Using this value for k, we can find the time t when $P = 5000$:

$$\begin{aligned} 5000 &= 10{,}000e^{\frac{\ln(0.8)}{3}t} \\ 0.5 &= e^{\frac{\ln(0.8)}{3}t} \\ \ln(0.5) &= \frac{\ln(0.8)}{3}t\ln(e) \\ \frac{3\ln(0.5)}{\ln(0.8)} &= t \approx 9.32 \text{ hr.} \end{aligned}$$

18. $a = 1/3$. The solution to question 14 contains an analysis of the graph of $|2x| + |2y| = 1$. We expect the graph of $|ax| + |ay| = 1$ to be a square with vertices on the x- and y-axes. Since the area of the square is 18 units^2, each diagonal will have length 6 units. The x- and y-intercepts must be ± 3. For $x, y \geq 0$, the

Coffee Type	Lbs.	Price per Lb.	Total Price
Kona	50	\$34.95	$50 \cdot \$34.95 = \$1747.50$
Colombian	x	\$7.99	$\$7.99x$
Mixture	$50 + x$	\$11.95	$\$11.95(50 + x)$

equation can be written as $ay = -ax + 1 \rightarrow y = -x + 1/a$. In quadrant I, the graph must coincide with the graph of $y = -x + 3$. Therefore, $1/a = 3 \rightarrow a = 1/3$.

19. 290.4 lbs. To best organize the information for this mixture problem, consider the chart shown on the previous page (p. 604, bottom). We see that the rightmost column gives us the information that we need to solve for the number of lbs. of Colombian beans required. Since the total cost of the mixture must equal the sum of the costs of the Kona and Colombian beans, we set up the following equation:

$$\begin{aligned} 1747.50 + 7.99x &= 11.95(50 + x) \\ 1747.50 + 7.99x &= 597.50 + 11.95x \\ 1150 &= 3.96x \\ x &= \frac{1150}{3.96} \approx 290.4 \text{ lb.} \end{aligned}$$

Barstucks must purchase 290.4 lbs. of Colombian beans to produce a Kona-Colombian blend worth $11.95 per lb.

20. 284,765,625 license plates. Using the fundamental counting principle, we multiply as follows:

$$25 \cdot 9 \cdot 25 \cdot 9 \cdot 9 \cdot 25 \cdot 25 = 9^3 25^4$$

Then $9^3 \cdot 25^4 = 284{,}765{,}625$.

21. Yes. Use the Pythagorean theorem to find the distance d from the tower to the house: $10^2 + 5^2 = d^2 \rightarrow 125 = d^2 \rightarrow d \approx 11.2 < 12$. The house lies within the signal range.

22. Charles—ribs; John—hamburgers; Kim—hot dogs; Lucy—pasta salad; Violet—veggie burgers. Match Lucy with pasta salad. The only other vegetarian dish is veggie burgers, so that must be Violet's favorite food. Kim likes neither ribs nor hamburgers; she must, therefore, like hot dogs. Since Charles does not like hamburgers, he must like ribs. Hamburgers remain for John to enjoy.

23. axy. A negative exponent indicates the reciprocal of a number: a^{-2} becomes $1/a^2$ when we use positive exponents. We eliminate the negative exponents and then multiply by the reciprocal of the third factor:

$$\left(\frac{b^2y^2y^3}{a^3x^2x^3}\right) \cdot \left(\frac{ax^2}{by^3}\right)^2 \cdot \frac{(axy)^2}{1}$$

Next, we perform the indicated multiplication and squaring:

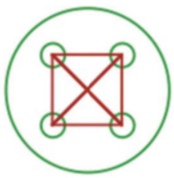
$$\left(\frac{b^2y^5}{a^3x^5}\right) \cdot \left(\frac{a^2x^4}{b^2y^6}\right) \cdot \frac{a^2x^2y^2}{1}$$

We multiply the factors in the numerators and the factors in the denominators, looking for variables that appear more than once:

$$\left(\frac{a^4b^2x^6y^7}{a^3b^2x^5y^6}\right) = axy$$

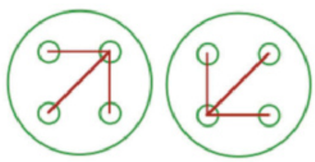
24. 42,711. We are told that the digits in the ones place and in the tens place are the same, so we can use the letters a, b, c, and d to represent the five-digit number: $abcdd$. The leftmost digit is twice the second, so $a = 2b$. The third digit is 1 more than the sum of the two to its left, so $c = a + b + 1 \rightarrow c = 3b + 1$. Since $c \le 9$, we have $3b + 1 \le 9 \rightarrow b \le 2\ 2/3$. So $b = 1$ or 2. Our number is either $427dd$ or $214dd$. The sum of the digits is 15, which means that $d = 1$ or $d = 4$. We reject the latter possibility because the fourth digit must be different from the digits to its left. The five-digit number 42,711 satisfies all the conditions.

25. 63 patterns. The button has four holes, any two of which can be connected with thread. As the figure below shows, the thread can make six distinct connections.



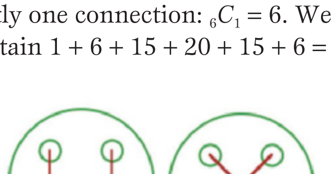
(a)

Once a button has been sewn to the garment, it can no longer be rotated; thus, the figure below shows two different patterns.



(b)

There is exactly one way to make the 6 connections: ${}_6C_6 = 1$. There are 6 ways to omit one of the connections; ${}_6C_5 = 6$. There are 15 ways to omit two of the connections: ${}_6C_4 = 15$. Two of the 20 ways to omit 3 of the connections are shown in the figure above: ${}_6C_3 = 20$. Two of the 15 ways to omit 4 of the connections are shown in the figure below: ${}_6C_2 = 15$. There are 6 ways to create exactly one connection: ${}_6C_1 = 6$. We add to obtain $1 + 6 + 15 + 20 + 15 + 6 = 63$.



(c)

Alternate solution: There are 2 possibilities for each of the thread connections: present or absent. Therefore, $2^6 = 64$ comes close to the answer. We cannot allow all connections to be absent, because at least one connection is required to attach the button to the garment. Thus, $2^6 - 1 = 63$.

26. $EC = 18$. Since $\overline{DE} \parallel \overline{BC}$, $\triangle ADE \sim \triangle ABC$. $DE = 6$, and $BC = 18$, so the scale factor is 1:3. The perimeter of $\triangle ADE = 81/3 = 27$. Since $AD = 12$ and $DE = 6$, we have $AE = 9$. The ratio of AE to AC = 1:3, so $AC = 27$ and $EC = 18$.

27. About 9.184 sec. Use the formula $v_t = v_{initial} + (acceleration)(time)$ or $v_t = v_0 + at$. We know that the shell is launched at $v_0 = 90$ m/sec. and that it will explode when its velocity is zero, or $v_t = 0$. We write the equation as follows:

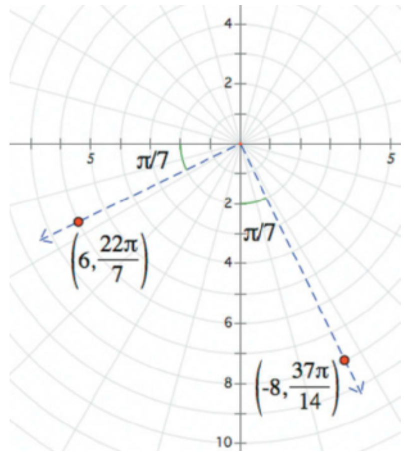
$$0 \text{ m/sec.} = 90 \text{ m/sec.} + \frac{-9.8 \text{ m/sec.}}{\text{sec.}}(t \text{ sec.})$$

Therefore,

$$t = \frac{-90}{-9.8}\text{sec.} \approx 9.184 \text{ sec.}$$

28. –5, 2, 3, 5, and 7. If $a^b = 1$ for real numbers a and b, then there are three possibilities: $b = 0$ and $a \neq 0$; or $a = 1$; or $a = (-1)$ and b is an even integer. Case 1: Let $x^2 - 2x - 35 = 0 \rightarrow (x - 7)(x + 5) = 0 \rightarrow x = 7$ or $x = -5$. For either value of x, $(x^2 - 7x + 11) \neq 0$. Case 2: Let $x^2 - 7x + 11 = 1 \rightarrow x^2 - 7x + 10 = 0 \rightarrow (x - 2) \cdot (x - 5) = 0 \rightarrow x = 2$ or $x = 5$. Case 3: Let $x^2 - 7x + 11 = -1 \rightarrow x^2 - 7x + 12 = 0 \rightarrow (x - 3)(x - 4) = 0 \rightarrow x = 3$ or $x = 4$. We must determine whether these x-values result in an even value for the exponent, $x^2 - 2x - 35$. If $x = 3$, then $x^2 - 2x - 35 = 3^2 - 2(3) - 35 = -32$, which is even. If $x = 4$, then $x^2 - 2x - 35 = 4^2 - 2(4) - 35 = -27$, which is odd. We reject $x = 4$.

29. 10 units. The point with polar coordinates $(6, 22\pi/7)$ is 6 units from the pole, or origin. Since $22\pi/7 = 3\pi + \pi/7$, we see that $(6, 22\pi/7)$ coincides with the point $(6, 8\pi/7)$. The point lies in quadrant III, on a line that makes an angle of $\pi/7$ with the ray opposite the polar axis. The point with coordinates $(-8, 37\pi/14)$ is 8 units from the pole. The negative sign requires us to measure the angle from the ray opposite the polar axis. Thus, $37\pi/14 = 2\pi + \pi/2 + \pi/7$; the point lies in quadrant IV on a line that makes an angle of $\pi/7$ with the vertical. Both points are shown in the figure. The points lie on two perpendicular rays, so the Pythagorean theorem gives us the familiar 6-8-10 triple.



30. 5/14. The sample space consists of ${}_8C_3 = 56$ equally likely combinations of 3 marbles. There are ${}_6C_3 = 20$ ways to select 3 of the 6 nongreen marbles. The probability that no green marble is selected is thus $20/56 = 5/14$.

Alternate solution 1: The probability that the first marble selected is not green is $6/8 = 3/4$. The probability that the second marble selected is not green, given that the first is not green, is 5/7. The probability that the third marble selected is not green, given that the previous selections are not green, is $4/6 = 2/3$. The probability of all three events is the product of the three: $3/4 \cdot 5/7 \cdot 2/3 = 5/14$.

Alternate solution 2: This approach sums all successful events: the 1 way to choose all red marbles; the 6 ways to choose 2 red marbles and 1 blue marble; the 3 ways to choose 2 red marbles and 1 white marble; the 3 ways to choose 1 red marble and 2 blue marbles; the 6 ways to choose 1 red, 1 white, and 1 blue marble; and the 1 way to choose 2 blue marbles and 1 white marble. Adding, $1 + 6 + 3 + 3 + 6 + 1 = 20$ provides the numerator, and ${}_8C_3$ is the denominator, yielding 5/14.