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Interesting solutions and ideas emerge when preservice and in-service teachers are asked a traditional algebra question in new ways.

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Sometimes, in the teaching and learning of mathematics, open-ended problems posed by teachers or students can lead to a fuller understanding of mathematical concepts—a depth of understanding that no one could have anticipated. Interesting solutions and ideas emerged unexpectedly when we asked prospective and in-service teachers an "old" algebra question in new ways. Our initial goal was to model NCTM's Process Standards in our classrooms (NCTM 2000). The result was a deeper understanding of solutions that emerge from the algorithm for solving equations involving radicals.

The old question was simply to find all solutions—true and extraneous—to radical equations. (We call equations containing radical expressions with variables in the radicand *radical equations.*) Almost every traditional high school algebra textbook includes one section devoted to solving equations of the form $\sqrt{ax+b} = cx + d$. Teachers usually instruct students to square both sides of the equation to eliminate the radical and to solve using basic algebra. Students are then told to check their answers in the original equation because extraneous solutions may appear. Often not discussed is what extraneous solutions really are and why they exist.

CHARACTERIZING SOLUTIONS OF A BASIC RADICAL EQUATION

The first new question was to characterize all possible solutions of the equation of the form $\sqrt{ax+b} = cx + d$. Goolsby and Polaski (1997) provide a graphical interpretation of the solution of $\sqrt{ax+b} = cx + d$ as the intersection of a radical function defined by the left side of the equation and a linear function defined by the right side of the equation. After some discussion of extraneous solutions, we were able to conclude that true solutions are found where the radical function intersects the line and that extraneous solutions are

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found where the reflection of the radical function (specifically, $y = -\sqrt{ax+b}$) intersects the line. The graphical approach provides an easy way to visualize all possible solution combinations. All possible solutions include case 1, with one true solution and one extraneous solution (see **fig. 1a**); case 2, with one true solution and no extraneous solutions (see **fig. 1b**); case 3, with one extraneous solution and no true solutions (see **fig. 1c**); case 4, with only imaginary solutions (see **fig. 1d**); case 5, with two true solutions (see **fig. 1e**); and case 6, with two extraneous solutions (see **fig. 1f**).

To characterize the solutions algebraically as well as graphically, students carried out the typical algorithm. Squaring both sides of the original equation ($\sqrt{ax+b} = cx + d$) left us with $c^2x^2 + (2cd - a)x + d^2 - b = 0$. The solution was

$$x = \frac{-2cd + a \pm \sqrt{a^2 - 4acd + 4c^2b}}{2c^2}.$$

Next, we considered the discriminant and generalized the types of solutions that we would obtain, depending on the values of $a, b, c,$ and d. If $a^2 - 4acd + 4c^2b < 0$, then we have no true or extraneous solutions (the algebraic formulation of case 4). If $a^2 - 4acd + 4c^2b = 0$, then the solution takes the form

$$x = \frac{-2cd + a}{2c^2}.$$

In this instance, we have either one true solution or one extraneous solution (cases 2 or 3).

More specifically, we can note that we would have an extraneous solution if

$$cx + d < 0 \rightarrow c\left(\frac{-2cd + a}{2c^2}\right) + d < 0 \rightarrow \frac{a}{2c} < 0$$

(case 3). Then, using a similar argument, we would have one true solution if $a/2c > 0$. Finally, if $a^2 - 4acd + 4c^2b > 0$, we have two solutions (cases 1, 5, and 6).

A NEW QUESTION INVOLVING RADICAL EQUATIONS

We then posed the following: Write a radical equation with a true solution at $x = 5$ and extraneous solutions at $x = -1$ and $x = -2$. Asking an open-ended question about these radical equations seemed like a logical next step. We were surprised and pleased to find that the problem was more difficult than we had expected and led to some interesting solution strategies.

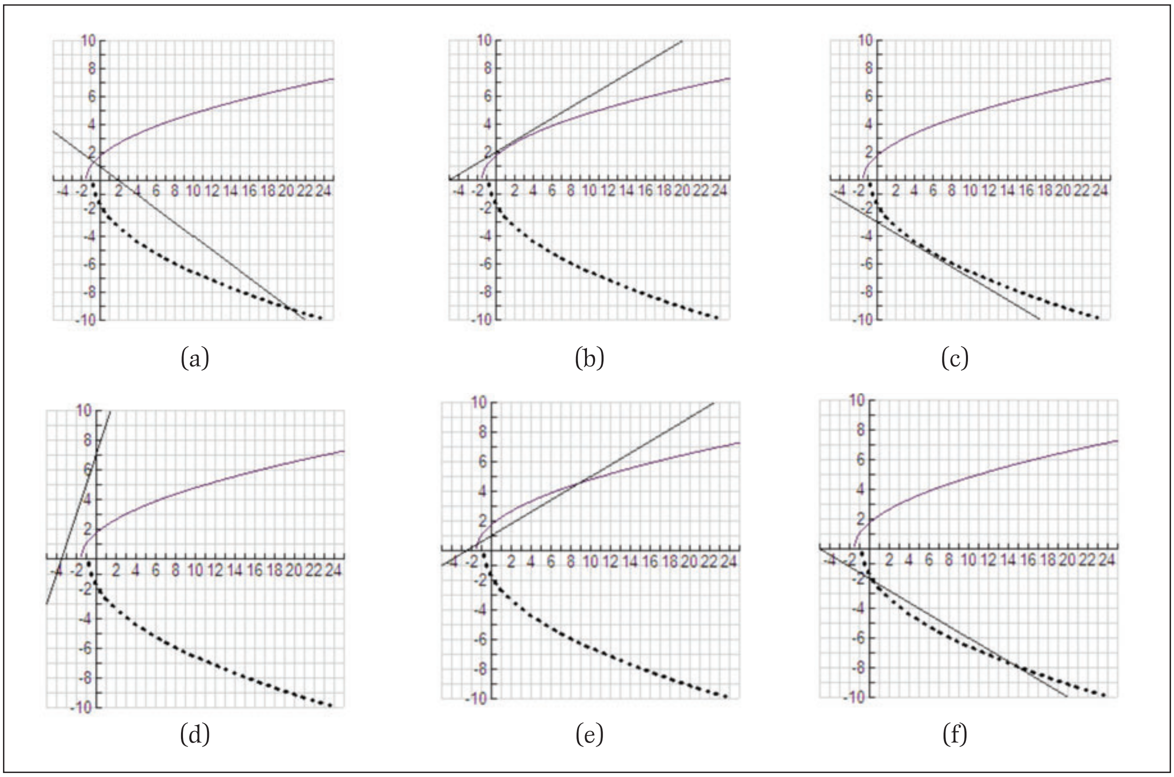


Fig. 1 These graphs (**a–f**) indicate all possible cases.

The new question led us to a discussion of the fundamental theorem of algebra and to a discussion of one-to-one functions. Note that an equation with three real solutions (extraneous or not) cannot be of the form $\sqrt{ax+b} = cx + d$ because the quadratic produced by squaring both sides of the equation has exactly two solutions. This is a basic application of the fundamental theorem of algebra.

Because we need three solutions, a typical first impulse in solving the new problem would be to consider a radical equation that would produce a cubic instead of a quadratic, an equation of the form $\sqrt[3]{ax+b} = cx + d$, for example. However, cubing both sides of this equation will not produce extraneous solutions because the basic cubic function $y = x^3$ is a one-to-one function.

Recall the definition of one-to-one functions: A function f is one-to-one if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ (where x_1 and x_2 are in the domain of f). When we solve any equation of the form $\sqrt{ax+b} = cx + d$, we can square both sides of the equation $f(\sqrt{ax+b}) = f(cx + d)$, but because the function is not one-to-one, we cannot then say that $\sqrt{ax+b} = cx + d$. When we solve any equation of the form $\sqrt[3]{ax+b} = cx + d$, we can cube both sides and be confident that $f(\sqrt[3]{ax+b}) = f(cx + d)$ does imply that $\sqrt[3]{ax+b} = cx + d$.

Method 1: Exploiting the Sign of the Extraneous Roots

We were thrilled to find that when students were unable to solve the problem, many used algebraic methods to solve a simpler problem that involved only two solutions, hoping that their technique could be generalized to the more difficult problem. They decided to find a radical equation that had one true solution at $x = 5$ and one extraneous solution at $x = -1$. Starting with the end in sight, $(x - 5) \cdot (x + 1) = 0$, which is equivalent to $x^2 - 4x - 5 = 0$, one student simply transformed this equation to $x = \sqrt{4x+5}$. Using technology, we can check for the solutions of this as we graph $y_1 = x$, $y_2 = \sqrt{4x+5}$ (which will show the true solution), and $y_3 = -\sqrt{4x+5}$ (which will show the extraneous solution).

This method was then extended to the problem at hand. Again, we started with the end in sight: $(x - 5)(x + 1)(x + 2) = 0$, which is equivalent to $x^3 - 2x^2 - 13x - 10 = 0$. We solved for

$$x^2 = \frac{x^3 - 13x - 10}{2}$$

so that we could take the square root of both sides, leaving

$$x = \pm\sqrt{0.5x^3 - 6.5x - 5}.$$

In graphing, we can see that this approach does, in fact, yield the desired solutions (see **fig. 2**).

Note that the method described here works only because the extraneous solutions we chose were negative; in other words, a negative value cannot be equal to a principal square root. What if we wanted all solutions to be positive? A well-chosen horizontal shift of both sides of the equation

$$x = \sqrt{0.5x^3 - 6.5x - 5}$$

would produce

$$x - 4 = \sqrt{0.5(x-4)^3 - 6.5(x-4) - 5},$$

an equation that has extraneous solutions at $x = 2$ and $x = 3$ and a true solution at $x = 9$.

Method 2: An Algebraic Approach

A second approach is similar to the first yet quite elegantly accounts for the negative extraneous solutions and the positive true solutions. The student who came up with this approach also attempted the simpler problem first (a true solution at $x = 5$ and one extraneous solution at $x = -1$). She carefully selected a perfect square trinomial $(2x - 3)^2 = 4x^2 - 12x + 9$ so that $2x - 3 < 0$ for $x = -1$ (the extraneous solution) and $2x - 3 > 0$ for $x = 5$ (the true solution). Next, she added the perfect square trinomial to both sides of the equation $x^2 - 4x - 5 = 0$ to obtain $5x^2 - 16x + 4 = (2x - 3)^2$. She then took the square root of both sides—

$$\sqrt{5x^2 - 16x + 4} = 2x - 3$$

—and achieved a radical equation with the desired solutions.

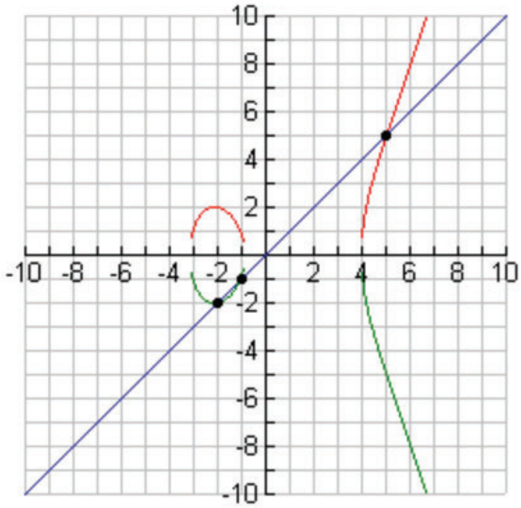
In extending this method to the problem with three solutions, we started with the same cubic as before: $x^3 - 2x^2 - 13x - 10 = 0$. We then carefully selected a perfect square, $(x - 3)^2$, to ensure that $x - 3 < 0$ for $x = -1$ and $x = -2$ (the extraneous solutions) and that $x - 3 > 0$ for $x = 5$ (the true solution). After adding the squared binomial to both sides of the equation, we had $x^3 - x^2 - 19x - 1 = (x - 3)^2$ and, finally,

$$\sqrt{x^3 - x^2 - 19x - 1} = x - 3.$$

Again, in graphing each side of this equation and the reflection of the radical in the x-axis, we had the desired solutions as the intersections.

Method 3: A Graphical Approach

A few students used a graphical approach for the simpler problem. They started with a function such as $y = \sqrt{x+2}$ and then determined a line through the points $(-1, -1)$ and $(5, \sqrt{7})$. These points were obtained using the same ideas as the previous two simpler solutions because we wanted the true solution to be at $x = 5$ and the extraneous solution at $x = -1$. Therefore, students using this method substituted $x = 5$ into their radical equation, producing



The graph of

$$y = x \text{ and}$$

$$y = \pm\sqrt{0.5x^3 - 6.5x - 5}$$

shows a true solution at $x = 5$ and extraneous solutions at $x = -1$ and $x = -2$.

Fig. 2 The true solution is at the intersection of the red curve with the line; extraneous solutions are at the intersection of the green curve with the line.

the value $\sqrt{7}$. When substituting $x = -1$, they got an output of $+1$; however, because this solution is meant to be the extraneous one, it would have an output of -1, according to the radical equation's reflection. The line passing through the two points is

$$y + 1 = \frac{\sqrt{7}+1}{6}(x+1),$$

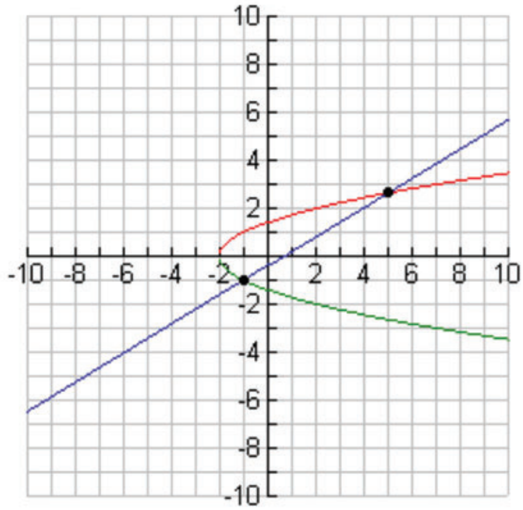
leading to the equation

$$\sqrt{x+2} = \frac{\sqrt{7}+1}{6}x + \frac{\sqrt{7}-5}{6}.$$

Thus, we arrive at the correct solutions (see **fig. 3**).

This method finally led us to a more general approach that will work for any number of true and extraneous solutions. This approach involved obtaining an equation of the form $f(x) = \pm\sqrt{g(x)}$ such that $f(-1) = -\sqrt{g(-1)}$, $f(-2) = -\sqrt{g(-2)}$ (to ensure extraneous solutions), and $f(5) = \sqrt{g(5)}$ (the true solution). The next step was to arbitrarily select "nice" values so that $g(-1) = 1$, $g(-2) = 4$, and $g(5) = 9$, leading to $f(-1) = -1$, $f(-2) = -2$, and $f(5) = 3$.

Therefore, we need a function g whose graph contains the points $(-1, 1)$, $(-2, 4)$, and $(5, 9)$ and a function f whose graph contains the points $(-1, -1)$, $(-2, -2)$, and $(5, 3)$. Because we had three points

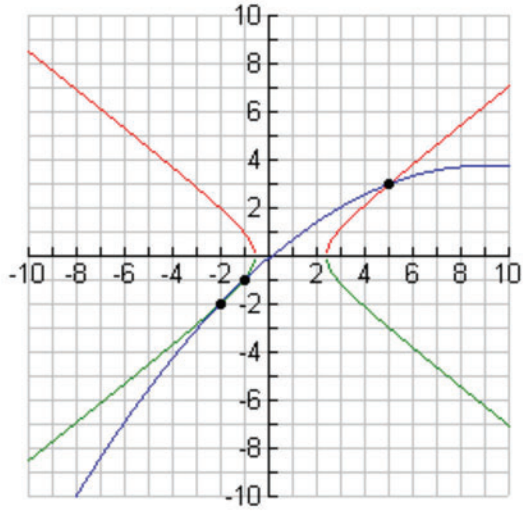


The graph of

$$\pm\sqrt{x+2} = \frac{\sqrt{7}+1}{6}x + \frac{\sqrt{7}-5}{6}$$

shows a true solution at $x = 5$ and extraneous solutions at $x = -1$.

Fig. 3 This graph shows a true solution at the intersection of the red curve and the blue curve and an extraneous solution at the intersection of the green curve and the blue curve.



The graph of

$$-\frac{1}{21}x^2 + \frac{6}{7}x - \frac{2}{21} = \pm\sqrt{\frac{13}{21}x^2 - \frac{8}{7}x - \frac{16}{21}}$$

shows a true solution at $x = 5$ and extraneous solutions at $x = -1$ and $x = -2$.

Fig. 4 This graph shows a true solution at the intersection of the red curve and the blue curve and extraneous solutions at the intersections of the green curve and the blue curve.

for each function, we concluded that we could calculate quadratic functions for $f(x)$ and $g(x)$.

We then substituted our points into $f(x) = ax^2 + bx + c$ to generate the two systems of equations below:

$$\begin{aligned} f(x) \to a(-1)^2 + {}& \\ b(-1) + c &= -1 \\ a(-2)^2 + b(-2) + c &= -2 \\ a(5)^2 + b(5) + c &= 3 \end{aligned} \qquad \begin{aligned} g(x) \to a(-1)^2 + {}& \\ b(-1) + c &= 1 \\ a(-2)^2 + b(-2) + c &= 4 \\ a(5)^2 + b(5) + c &= 9 \end{aligned}$$

We obtained the following functions—

$$f(x) = -\frac{1}{21}x^2 + \frac{6}{7}x - \frac{2}{21}$$

and

$$g(x) = \frac{13}{21}x^2 - \frac{8}{7}x - \frac{16}{21},$$

—leading to the equation

$$-\frac{1}{21}x^2 + \frac{6}{7}x - \frac{2}{21} = \pm\sqrt{\frac{13}{21}x^2 - \frac{8}{7}x - \frac{16}{21}}.$$

Graphically, we can see that the solutions are indeed those that we were looking for (see **fig. 4**).

Exploiting a Different Source of Extraneous Roots

Another group of students had been exploring equations such as

$$\frac{(x^2 + x - 2)}{x+2} = 4.$$

Applying the usual algorithm for solving the equation produces the solutions $x = -2$, which is clearly an extraneous solution, and $x = 5$. Note that $x + 2$ is a factor of the numerator and the denominator of the left side of the equation. To produce a true solution at $x = 5$, we begin with $x - 1 = 4$. To produce extraneous solutions at $x = -1$ and $x = -2$, we then form a rational expression on the left side of the equation:

$$\frac{(x-1)(x+1)(x+2)}{(x+1)(x+2)} = 4$$

This time the extraneous roots come from holes in the domain of the rational expression on the left side of the equation.

As a class, we also discussed other types of functions that are not one-to-one and that could therefore produce extraneous solutions when applied to both sides of an equation—trigonometric functions, for example.

CONCLUSION

Asking open-ended questions about an old problem led to a surprising variety of solutions and solution strategies that involved algebraic, tabular, and graphical representations of functions. The characterization of the solutions of the equation $\sqrt{ax+b} = cx + d$ illustrates the power of a graphical approach, whereas the algebraic approach makes specific the relationship among the parameters a, b, c, and d. Asking students to produce an equation with certain extraneous and true solutions led to a synthesis of many concepts, including the fundamental theorem of algebra, one-to-one functions, the problem-solving strategy of solving a simpler problem, horizontal shifts of functions, solutions of systems of equations, and functions with holes in their domains. Most important, we were able to engage students in an exploration of radical equations with true and extraneous solutions without simply relying on an algorithm that they had been taught. These investigations ultimately led them to a deeper understanding of the mathematical concepts.

As teachers, we gained new insights from students' ideas and strategies. Classroom discussions can be rich and powerful when students are encouraged to explore mathematics.

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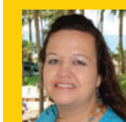

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