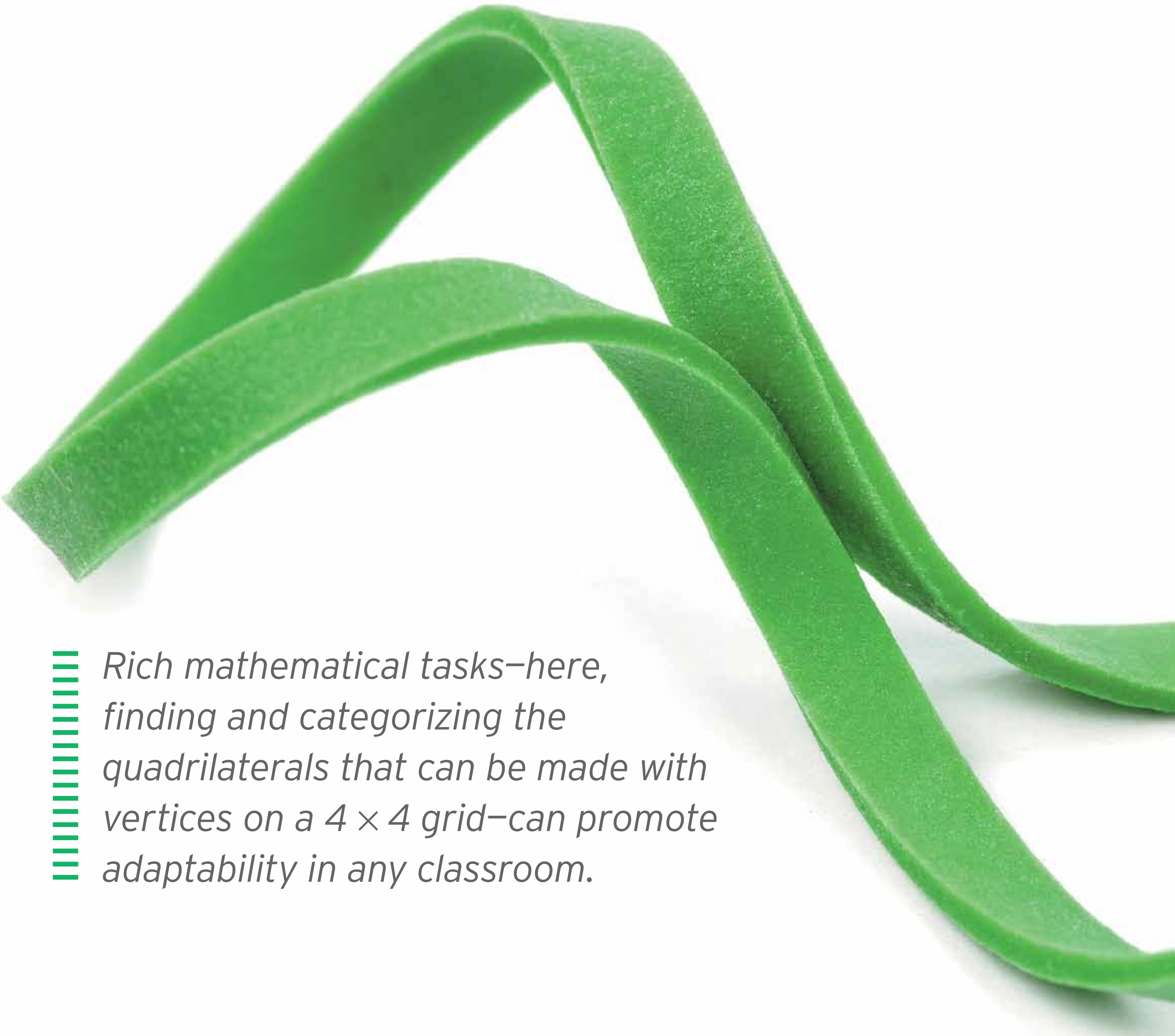
Exploring Quadrilaterals

Flexible A



Rich mathematical tasks—here, finding and categorizing the quadrilaterals that can be made with vertices on a 4 × 4 grid—can promote adaptability in any classroom.

through

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pproaches

The quadrilaterals problem is a particularly rich task, applicable to in-service teachers as well as to the students they teach:

> Given the set of sixteen dots shown (see **fig. 1**), draw all possible four-sided figures with corners at the dots. Develop a system for knowing when you have identified them all. Classify the quadrilaterals in as many ways as you can. Example: Classify by the number of right angles. What are the possible perimeters? What are the possible angle measures?

The ways in which students engage in this task, the strategies they use, and the mathematics that results align well with this *MT* Focus Issue on flexible mathematical thinking. We offer our observations of various groups of in-service teachers as they investigate this problem and then extend these insights by showing how middle-grades students approached the same task. The purpose of sharing these experiences is to expose the rich potential of the task and to highlight the flexible nature of students' thinking about the task.

The ideas presented here grew out of a presentation at an NCTM annual meeting and an article written from a theoretical perspective (Richardson, Reynolds, and Stein 2008; Richardson, Schwartz, and Reynolds 2010). We have also benefited from the works of Kennedy (1993), who looked at triangles on an array, and Kennedy and McDowell (1998), who provided the actual number of quadrilaterals on a 4 × 4 array.

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THE STUDENTS

Our graduate-level mathematics education students are predominantly a combination of in-service elementary, middle, and high school teachers. They often take the quadrilaterals task to their own classrooms and use it productively with their students, demonstrating that the task is suitable for students at various levels. In this on-going investigation, in-service teachers work alone or in groups of two, both in and outside class. At the end of the semester, pairs or small groups of teachers present their solutions to the entire class, usually in the form of PowerPoint® presentations but sometimes in other forms, such as large sheets of paper with various illustrations. The teachers are asked to describe and discuss the strategies that they used and developed over the course of the semester. What follows is a look at some of the solutions presented by in-service teachers and some discussion of how their own students investigated the task.

THE QUADRILATERALS PROBLEM

Task and Goal Clarification

What counts as an acceptable solution? One of the first questions posed by in-service teachers is, "Does a different location on the array count as a different solution?" (see **fig. 2**). The consensus is that it does not. Students must focus on two major goals of the problem: (1) to identify and count all *distinct* quadrilaterals that can be formed on a 4 × 4 grid and (2) to classify each quadrilateral they discover.

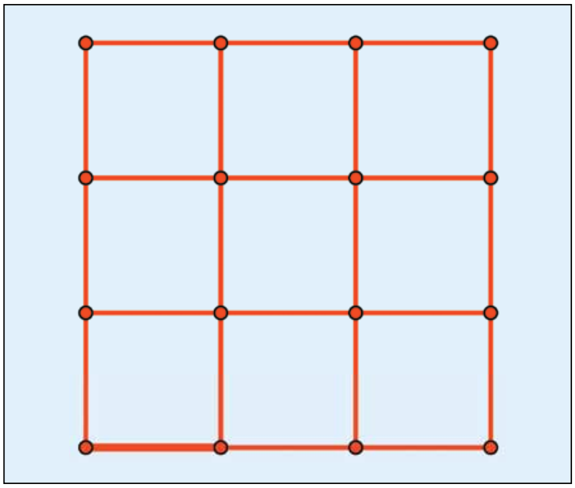


Fig. 1 How many quadrilaterals can be found?

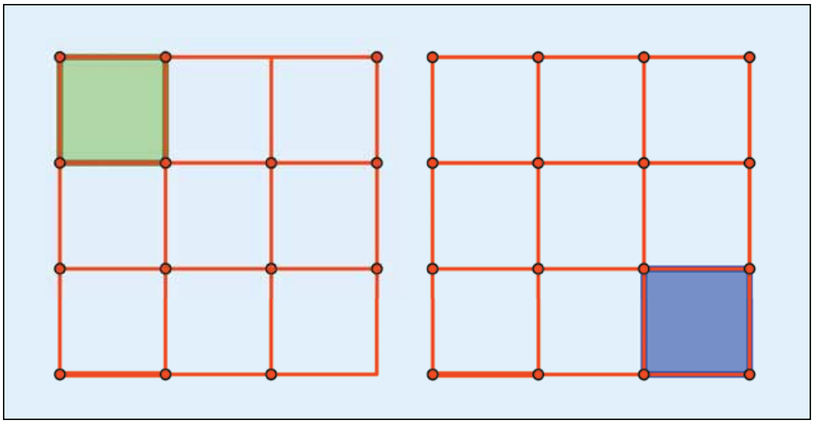


Fig. 2 A decision had to be made—one or two quadrilaterals?

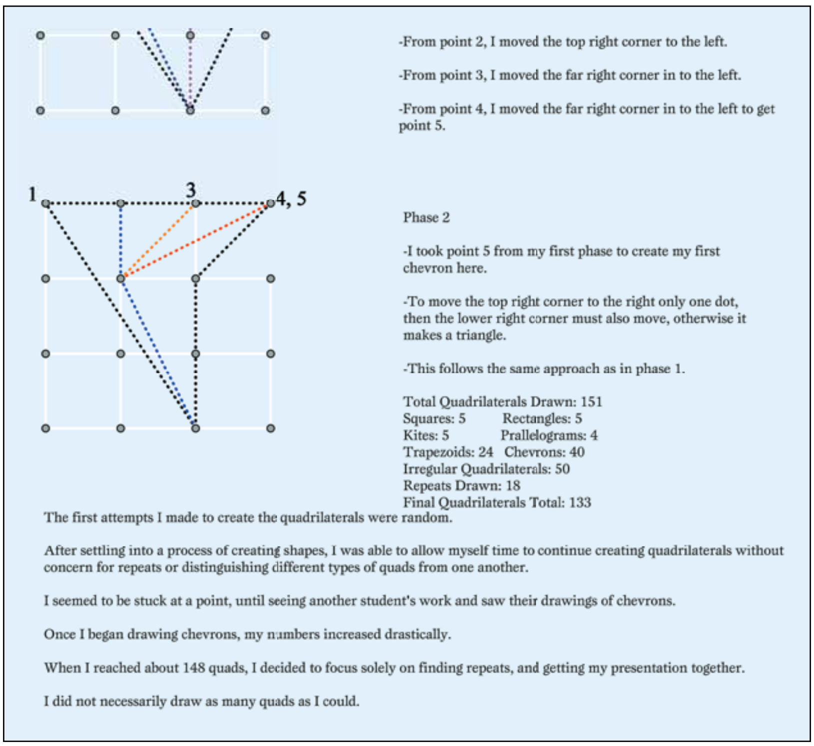


Fig. 3 This work sample provides a glimpse into this teacher's thinking as she progressed through the investigation.

Organization of Materials

A second common concern is how to organize the quadrilaterals. Important materials to have on hand include dot paper, geoboards, and computers, if available. Online geoboards that allow groups to place virtual rubber bands have proven to be quite helpful as a starting point or even as a way to check various quadrilateral duplicates efficiently. For many in-service teachers, the focus on developing a systematic way to keep track of the quadrilaterals is vital. The idea that a formula could be found for the number of all possible quadrilaterals emerges quite soon for the majority of students. Interestingly enough, however, these early starting points are almost always abandoned in pursuit of a more comprehensive or efficient approach.

In the following sections, we present work samples from our in-service teachers. Previously we have detailed similar ideas but from philosophical perspectives (see Richardson, Schwartz, and Reynolds 2010).

Orientation Strategies

Orientation of the quadrilaterals by shape on the array is the primary approach used by our students. One high school teacher developed a system of orienting the quadrilateral using three fixed points (see **fig. 3**) and then considered all the different types of quadrilaterals constructed using these points. As a result of engaging with others in the class about differing approaches, she realized that chevrons also needed to be a part of the quadrilaterals.

Another orientation strategy involved simultaneously fixing the number of dots or pegs and then systematically constructing all possible quadrilaterals with the selected dots or pegs. The example shown in **figure 4**, which was completed by two high school teachers, demonstrates that as the number of dots in the interior and on the perimeter of the quadrilateral increases, the variety and number of quadrilaterals also increase. A constraint of this approach relates to which fixed dots are selected in each case; selection appears to make a difference when counting distinct quadrilaterals.

A refinement of the approach shown in **figure 3** involved organizing the search by rows. **Figure 5**, which was completed by a middle-grades teacher, illustrates the beginning of this process using dots in the first two rows as anchors for the quadrilaterals. The process would be completed by using the first three rows and then all rows.

Some in-service teachers stay with their orientation strategy and see it through until they have exhausted what they believe to be all the quadrilaterals. Part of such a process involves noticing the duplicates that emerge. When teachers observe duplicates, the problem within the problem frequently emerges. Their attention then shifts to

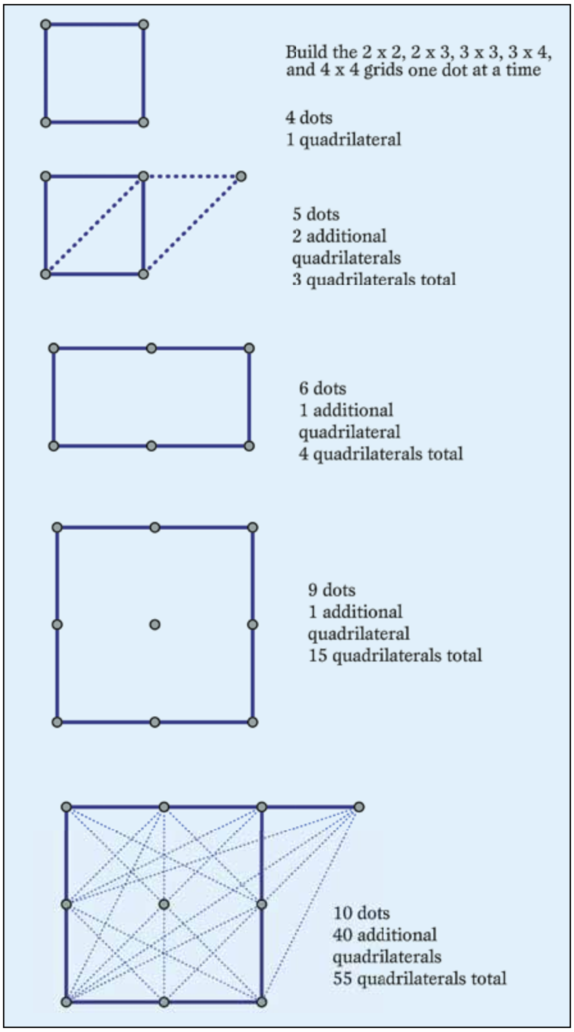


Fig. 4 Some students built up the complexity one step at a time.

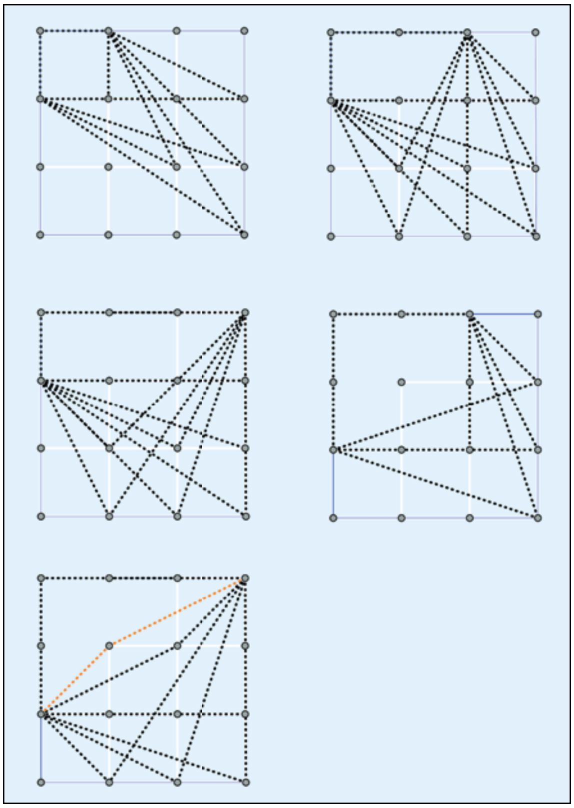


Fig. 5 One teacher used points and rows to anchor sets of quadrilaterals.

eliminating duplicates rather than generating new quadrilaterals. The process for deleting duplicates has involved strategies such as laying clear overhead projection paper over dot paper, tracing the quadrilateral, and then rotating the paper to see whether the traced quadrilateral is the same as another one.

The tedium of this strategy drives some teachers to move on to other approaches, such as focusing on the area and perimeter of the shapes that emerge. Thus, the challenge of identifying and tracking duplicates leads to a multipronged approach that uses various orientation strategies, along with area and perimeter of the quadrilaterals. This is the point at which the immense flexibility between approaches becomes clear.

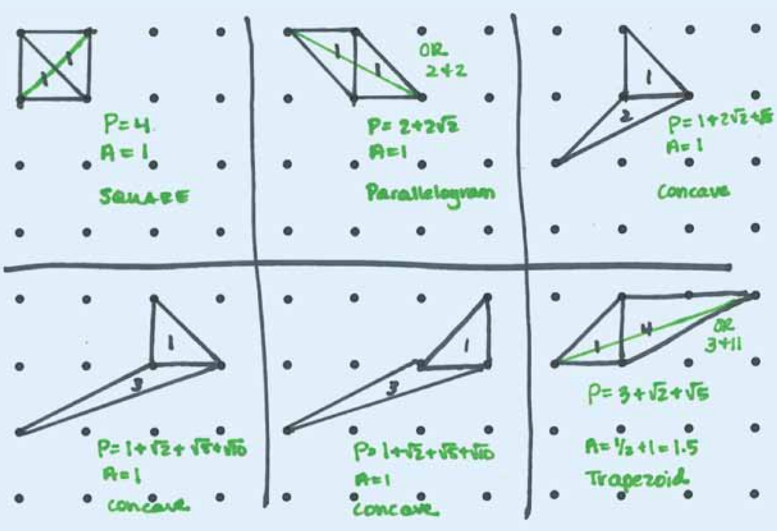
Measurement Strategies

Two other high school teachers who worked together used a perimeter approach. They began by identifying and documenting the types of quadrilaterals they found. Their knowledge of the Pythagorean theorem, however, moved their focus to the length of the sides of the quadrilaterals, and they came up with numbers involving square roots for possible side lengths and perimeters. The exploration became a more specific investigation of the perimeter of the quadrilaterals.

Another in-service high school teacher demonstrated a combination of orientation, perimeter, and area strategies (see **fig. 6a**). This teacher allowed the orientation strategy to drive her work, but she soon started analyzing the sides and areas of shapes. She began with triangles because she found them to be more manageable (again, we see evidence of the flexibility of the task) and went on to use the triangles to form quadrilaterals, as shown in the work sample. Her presentation was on a large poster board; **figure 6b** shows some processes that she used. Again, we can see how the very nature of the investigation took a variety of twists and turns for this teacher and how the focus, rather than the final answer, became the approach or the strategy. Another possible strategy (one that we did not observe in use) might be combining quadrilateral classification with both area and perimeter computations to demonstrate uniqueness.

Similarities and Differences between Approaches

In-service teachers almost always identify quadrilaterals by shape. The next step is to attempt a classification that leads from small to larger



(a)

Where Have I Been?

- First approach was very straightforward: find all quads systematically.
- Then I noticed that the quads were in fact just triangles.
- Before finding all possible triangles, I spent several weeks trying to find an equation whose answer would answer the question, "How many quads are there?"
- Painstakingly I have set out to find all combinations of triangles that form a quadrilateral.

Where Can I Go?

- My first instinct would be to first recreate the entire process again. If I arrive at the same answer twice, then I would be more confident in my answer.
- Next, I could reorganize all the quads by their area and the triangles that are being used.
- Or I could reorganize all the quads by their most specific title. Then I could look for patterns with the type of quad and the triangles that are being used.

(b)

Fig. 6 One teacher used triangles to organize her approach (**a**) and her presentation (**b**).

quadrilaterals. Materials such as geoboards and transparency paper help. Generally, however, differences begin to emerge as a result of the participants' mathematical backgrounds.

For many K–8 teachers, orientation strategies become the primary focus of the task. For 9–12 teachers, the measurement strategies require knowledge of the Pythagorean theorem as well as the ability to work with irrational numbers; thus, they may use measurement strategies later, when they exhaust the usefulness of the orientation strategies. Both measurement and orientation strategies may also be used in conjunction with one another.

Unexpected Approaches

On two occasions, we have been surprised by unique approaches to the task.

In one class, a cooperating pair of teachers who had generated a large number of quadrilaterals were having difficulty sorting through them to identify and eliminate duplicates. None of the methods that they tried proved satisfactory. The teachers, knowing that vectors had something to do with directed line segments, surmised that they might use them as a way to identify duplicates. Although they were never able to make vectors work for them in their analysis of the various quadrilaterals, during subsequent weeks they learned a considerable amount about something that was for them a whole new field of mathematics—vector analysis.

In another class, two doctoral students took another noteworthy approach inspired by a statistics course they were taking together at the time. They connected their experiences in the class with the degrees-of-freedom concept (see **fig. 7**). If two of the sides of a quadrilateral were set, they reasoned, once they drew the third side, the fourth side would be determined. Therefore, their strategy was to draw all the possible quadrilaterals when two adjacent sides were fixed.

SHARING THE QUADRILATERALS TASK WITH MIDDLE-GRADES STUDENTS

When our in-service teachers have used this task with their students, they have found that these students can explore the task at different levels, depending on their mathematical development.

For students in grades 4–8, the quadrilaterals task connects well with much of the mathematics that they are just making sense of, particularly in their geometry, measurement, and number work. With younger students, the activity has encouraged the exploration of the different types of quadrilaterals. The students quickly find the regular quadrilaterals (squares) and then all possible rectangles and then branch out from there to other classes of quadrilaterals. An interesting result is that the shapes generated are usually all convex quadrilaterals. Here the geoboards that some students use become useful. As students stretch their rubber bands around pegs, some get caught unintentionally on other pegs, resulting in a quadrilateral that is not convex; should it count? After some discussion, the class agrees that indeed it should count, and the hunt begins for these "odd" shapes.

One pair of students decided to classify their quadrilaterals by the number of right angles formed. They began by getting five fresh sheets of dot paper and giving them headings such as these: "0 right ∠s," "1 right ∠," "2 right ∠s," "3 right ∠s," "4 right ∠s" (meaning exactly 0, 1, 2, 3, or 4 right angles).

They then began to organize their shapes, starting with the shapes they had found with no right angles. When the students reached the sheet headed "3 right ∠s," they were puzzled because they could not find any shapes to meet this criterion. They experimented for some time and decided that every quadrilateral that had three right angles had to have four. This conjecture led to an interesting class discussion about angle measure in quadrilaterals.

In trying to find the area of some of their quadrilaterals, students have to identify a unit, because none is specified. Usually, they define a *unit* as the distance between horizontally or vertically adjacent points on the grid. This definition leads to an interesting dilemma because students often label the distance between two adjacent points along a diagonal as 1 unit. Others go straight to labeling a square formed by joining four adjacent points as a "square unit." When they connect diagonally adjacent points to make a square, they frequently claim that it is also a square unit (see **fig. 8**). At this point, some groups of students are cutting out the shapes they had drawn on the square grid paper to categorize them and also to check for duplicates (by placing shapes on top of one another). Then they notice that these shapes do not have the same perimeter or area.

With one group of sixth-grade students, the in-service teacher encouraged them to find the length of the side of this square made on the diagonal. Students used the side of the unit square to measure off this length and found that it was more than 1 unit but less than 1 1/2 units; further investigation led them to find that it was between 1 1/4 and 1 1/2 units and then between 1 1/3 and 1 1/2 units.

At this point, students asked how they could get a more "exact" measurement. The teacher took the opportunity to introduce the Pythagorean theorem to the class, and students used class computers to explore visual animations of various proofs of this theorem. They then began forming right triangles to calculate various diagonal lengths on the 16-point grid. They found that the length of this "diagonal square" (as they called it) was $\sqrt{2} \approx 1.4142$. To check this result against their earlier prediction that the length was between 1 1/3 and 1 1/2, they converted these fractions into decimals—1.33 and 1.5—and convinced themselves that their original prediction was good. For a diagonal length that extended over three dots, some students measured this as $2\sqrt{2}$ (reasoning that connecting two dots diagonally was $\sqrt{2}$ and that a segment connecting three dots was just double that length), whereas other students used the Pythagorean theorem to find this length to be $\sqrt{8}$, or 2.8284. The teacher challenged students to convince themselves that $2\sqrt{2}$ and $\sqrt{8}$ were the same without using decimals. This challenge led to an exploration of how to simplify square roots.

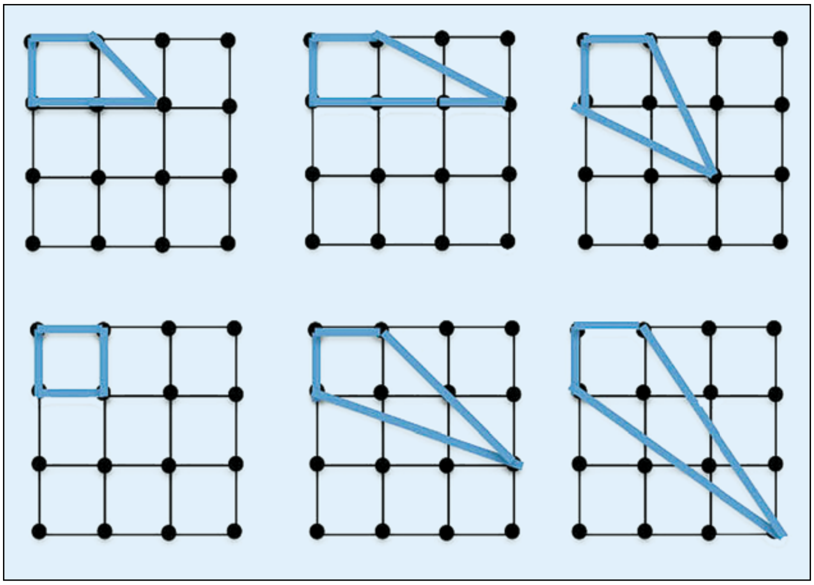


Fig. 7 Students taking a statistics course contemporaneously found a unique approach.

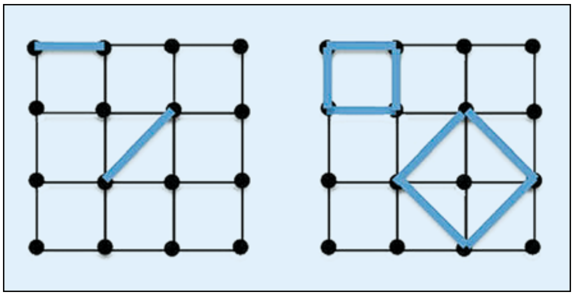


Fig. 8 Some students thought that both segments were units and that both squares were unit squares.

DISCUSSION

The quadrilaterals task has proven to be rich in mathematical ideas. In-service teachers used both geometric and measurement concepts in various problem-solving strategies in their attempts to solve this problem. The geometric approach often results in an exploration of spatial concepts, angles, transformations, congruence and similarity, and concave and convex shapes. The measurement approach frequently takes into account variable concepts, area and perimeter, the Pythagorean theorem, linear and square units, and rational and irrational numbers (to name a few).

With respect to using all strategies or viewing the similarities of other solutions, teams or individual students generally do not use all strategies but almost always share their work as the investigation progresses. This sharing allows the in-service teachers to appreciate one another's thinking and gain additional insight into the mathematics. The solutions in essence provide a means for organizing the same information (for example, students may look at two quadrilaterals and intuitively guess that one has greater perimeter than the other, or perhaps the

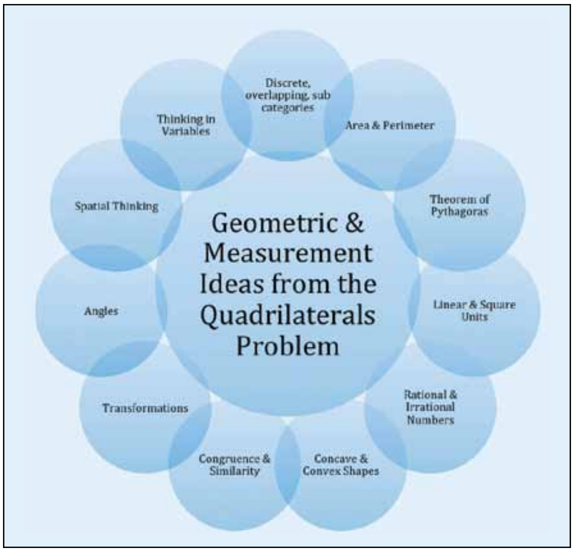


Fig. 9 The quadrilaterals task encompasses a great deal of mathematics, both on the surface and below it.

teacher can also provide string and students can use that to measure perimeter, thus avoiding the irrational number computations). When in-service teachers were faced with a challenge (i.e., looking at two quadrilaterals that looked as if they had the same perimeter), they tried to come up with a resolution. They might consider this question: Is there a way to get an accurate measure of the size? This question then leads to the Pythagorean theorem.

Figure 9 may be a useful tool for a teacher who is planning the activity in class because it conveys the fluidity of both approaches, which is important in representing all student experiences with the problem. All students need this type of flexibility when investigating such a rich task because they often move between and combine multiple approaches. In addition, there is an opportunity to discuss with students that this is what mathematicians do all the time—explore new ideas, make conjectures, argue about their latest findings, and formalize their work.

As the mathematician John H. Conway stated on the *Nova* episode devoted to Andrew Wiles's proof of Fermat's last theorem (Lynch and Singh 1997): "I'm sad in some ways, because Fermat's last theorem has been responsible for so much. What will we find to take its place?" Connected tasks (Richardson, Carter, and Berenson 2010) such as this quadrilaterals investigation give prospective and practicing teachers a chance to experience for themselves the value of learning mathematics through problem solving (Hiebert 1999; Wheatley 1991).

ACKNOWLEDGMENTS
The authors wish to thank Preety Tripathi of the Mathematics Department at SUNY at Oswego for her help in editing this article as well as all the students who worked so diligently on this problem.

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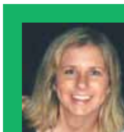


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