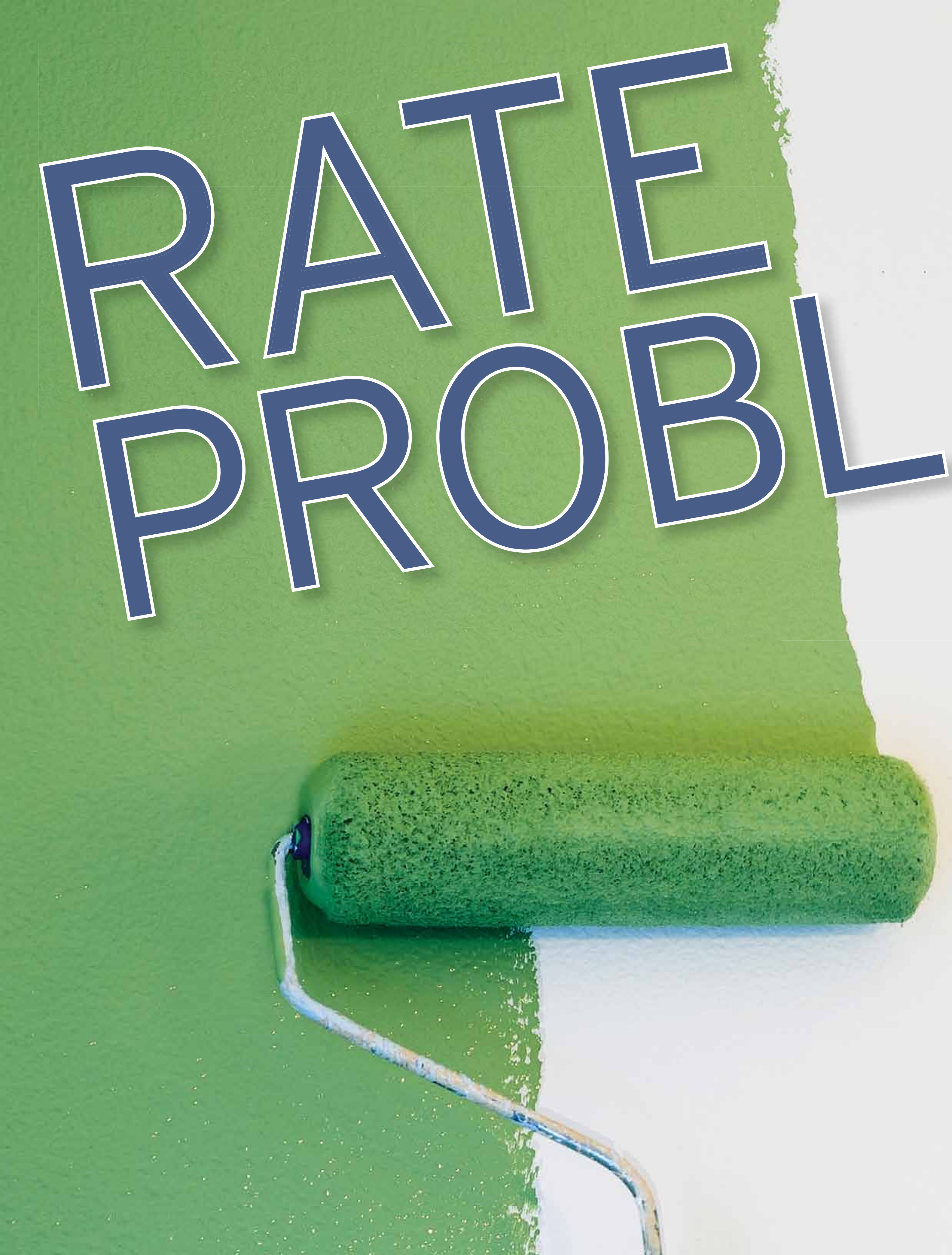
RATE PROBL



EMS: Thinking across the Curriculum

In a new approach, rate problems—the bane of students—can connect to higher-level concepts.

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An important concept in mathematics, yet one that is often elusive for students, is the concept of rate. For many real-life situations—those involving work, distance and speed, interest, and density—reasoning by using rate can be an efficient strategy for problem solving.

Students struggle with the concept of rate, despite the many possible applications. According to NCTM's Algebra Standard, students are expected to "model and solve contextualized problems using various representations, such as graphs, tables, and equations" by the end of grade 8 and "identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships" by the end of grade 12 (NCTM 2000, p. 395). We present an alternative approach to rate problems and show some connections between these problems and concepts taught in higher-level mathematics. An approach that makes use of functions provides students with a more robust understanding of mathematics, particularly because this approach facilitates multiple solution paths to the same problem.

A TYPICAL TEXTBOOK RATE PROBLEM

Consider the following problem, found in a typical first-year algebra textbook:

> Janice takes 4 hours to paint a room. Kathleen takes 5 hours to do the same job. How long would it take them, working together, to paint the room?

Typical strategies for solving this problem are based on proportional reasoning, algebraic reasoning, or modeling using fractions. Typical textbook examples of the latter two strategies follow.

Algebraic Reasoning

Let x = the time needed working together. Then

$$\begin{aligned} \frac{1}{4}x + \frac{1}{5}x &= 1 \\ 5x + 4x &= 20 \\ 9x &= 20 \\ x &= \frac{20}{9} = 2\frac{2}{9} \text{ hr.} \end{aligned}$$

Using Fractions

Janice paints 1/4 of a room in 1 hour. Kathleen paints 1/5 of the room in 1 hour. Together they paint 1/4 + 1/5 = 9/20 of the room in 1 hour. So the question becomes, How many 9/20 are there in 20/20? The answer is 2 2/9 (see **fig. 1**). (See Shore and Pascal [2008] for further discussion of these strategies.)

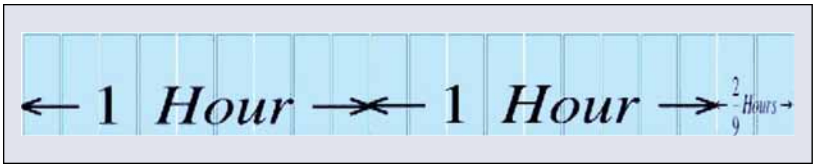

Fig. 1 How many 9/20s are there in 20/20s?

ALTERNATIVE STRATEGIES AND THEIR LINKS TO HIGHER-LEVEL MATHEMATICS

The same problem can be solved using either rate functions or work functions, as shown below.

Rate Functions

Define functions $R_K(t)$ and $R_J(t)$ to represent the rates for Kathleen and Janice, respectively. Because Kathleen and Janice are painting at a constant rate, $R_K(t)$ and $R_J(t)$ are constant functions. Specifically, $R_K(t) = 1/5$, $t \geq 0$, and $R_J(t) = 1/4$, $t \geq 0$.

If Kathleen and Janice work together, let their combined rate of painting be $R_{K\&J}(t) = R_K(t) + R_J(t) = 9/20$, $t \geq 0$. **Figure 2** shows the graphs of these functions. From these graphs, we observe that Janice paints faster than Kathleen, that their combined rate is faster than either of them working alone, and that their combined rate is also constant.

From the equation work = rate • time, it follows that the amount of time required to paint 1 room by Janice (t_J), by Kathleen (t_K), and by working together ($t_{K\&J}$) can be found by solving the equations: $(1/4)t_J = 1$, $(1/5)t_K = 1$, and $(9/20)t_{K\&J} = 1$.

Work Functions

Another way of thinking about this problem is by using functions of work done. Using the equation work = rate • time, we define work functions for Kathleen, Janice, and both of them working together as, respectively, $W_K(t) = (1/5)t$, $t \geq 0$;

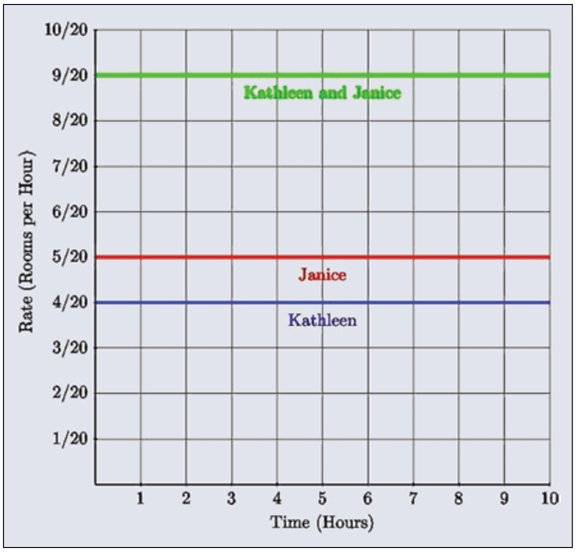

Fig. 2 Rate functions $R_K(t)$, $R_J(t)$, and $R_{K\&J}(t)$ can be graphed to show comparative speeds.

$W_J(t) = (1/4)t$, $t \geq 0$; and $W_{K\&J}(t) = (9/20)t$, $t \geq 0$ (see **fig. 3**).

Some connections can be made between these work functions and their corresponding rate functions (see **fig. 2**). When we use this approach to explore rate problems with teachers and students, we ask the following types of questions to stimulate their thinking about the graphs and the functions they represent:

- What is the significance of the linearity of the graphs?
- What do the slopes of these functions represent?
- What would be realistic domains for these functions?
- How could these graphs be used to solve the problem?

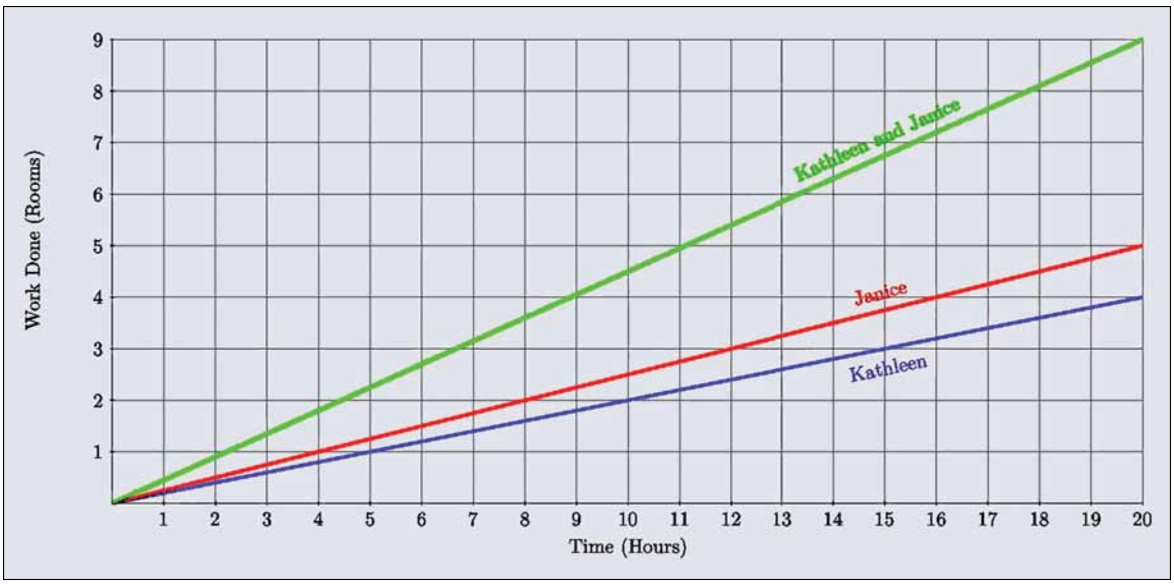

Fig. 3 Graphs of the work functions $W_K(t)$, $W_J(t)$, and $W_{K\&J}(t)$ provide a different representation for comparisons of the painters.

Through questioning, we guide explorations of the graphs and elicit meaningful discussions of the problem. By observing that the line $y = 1$ represents 1 room painted, we can develop a rich response to the fourth question above. We focus on the graph of the two painters working together, $W_{K\&J}(t)$, noting that it crosses $y = 1$ farther to the left than does either of the other two graphs.

Some additional questions specific to the graph of $W_{K\&J}(t)$ that we have asked to stir students' thinking about these graphs of work functions include these:

- How do the ordered pairs $(t, W_K(t))$ and $(t, W_J(t))$ relate to $(t, W_{K\&J}(t))$? What does this relationship mean in the context of the painters?
- How does the slope of $W_{K\&J}(t)$ compare with the slopes of $W_K(t)$ and $W_J(t)$? What does this comparison mean in context?
- Using the graph of $W_{K\&J}(t)$, estimate how long it would take Janice and Kathleen, working together, to paint 1 room, 2 rooms, and n rooms.

For these rates of Kathleen and Janice working individually, an exact solution to the problem is hard to read from **figure 3**. However, by asking for an estimate, we help students define a small interval within which the exact value is contained and simultaneously build on the important problem-solving strategy of estimating from a graph.

To find the exact time needed by Janice and Kathleen, reconsider the graphs in **figure 3** over a longer period of time. From the graphs, we can see that the first time both Janice and Kathleen will have painted a whole number of rooms (5 and 4, respectively) is after 20 hours. At that point, their combined work is 9 rooms. Thus, on average, they need 20/9 hours per room. These approaches create opportunities for a richer discussion about rate rather than by merely setting up the typical equation: $r_1x + r_2x = 1$. In addition, we begin to build a depth of understanding for the concept of function, known to be slow to develop and difficult for students to master (Carlson 1998).

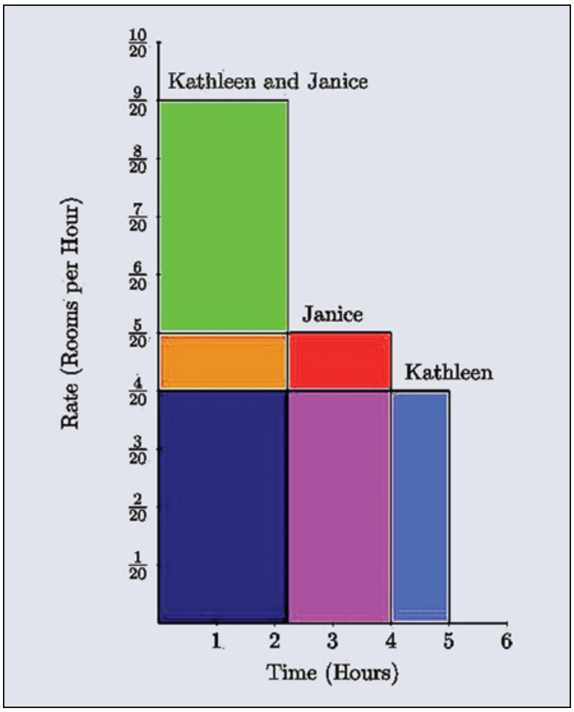

Fig. 4 This graph connects area to work.

CONNECTIONS TO CALCULUS CONCEPTS

This functional approach to solving rate problems creates opportunities for teachers to begin building a foundation for calculus concepts. For example, from **figure 2,** the values t_J, t_K, and $t_{K\&J}$ that satisfy the equations $(1/4)t_J = 1$, $(1/5)t_K = 1$, and $(9/20)t_{K\&J} = 1$, respectively, are the values that would make the area under the corresponding graph 1 unit (see **fig. 4**). In calculus, the question, What value of $t_{K\&J}$ will make the area under the function $R_{K\&J}(t)$ equal to 1? is equivalent to asking students to solve the following problem:

$$\int_0^{T_{K\&J}} \frac{9}{20}dt = 1$$

This functional approach can also be used to highlight the concepts of sums of functions and sums of derivatives, which can be achieved without necessarily using calculus terminology. Further, this approach highlights the invariance of the area representing one painted room across the three rate cases. It follows that the greater the rate (the height of the rectangle), the smaller the time (the width of the rectangle) needed to make the area under the rectangle 1 unit. Thus, the rate of painting is inversely proportional to the time needed to paint a room. This inverse relationship—

$$\text{rate (height)} = \frac{1}{\text{time (width)}}$$

—can be "seen" as a curve connecting the top right vertices of the rectangles (see **fig. 4**).

INCREASING THE CHALLENGE

Mathematics educators have made a strong case for careful consideration of the tasks that teachers use during instruction. Lappan and Briars (1995) argue that "there is no decision teachers make that has a greater impact on students' opportunities to learn and on their perceptions about what mathematics is than the selection or creation of the tasks with which the teacher engages students" (p. 138). We seek to engage students in explorations of rate that challenge their thinking and go beyond the rote manipulation of equations.

Consider the following problems:

> Two candles of equal length are lit at the same time. One candle takes 9 hours to burn out, and the other takes 6 hours to burn out. If both candles burn at a constant rate, after how much time will the slower-burning candle be exactly twice as long as the faster-burning one? (Kroll, Masingila, and Mau 1992)

> A dirt biker must circle a 5-mile track twice. His average speed must be 40 mph. On his first lap, he averaged 25 mph. How fast must he travel during his second lap to qualify? (Musser, Burger, and Peterson 2008)

These two problems ask the students to compare the amount of work still to be done (the candle problem) and average rate of change to instantaneous rate of change (the biker problem). We argue that both these problems have higher cognitive demands than those of the painters' problem presented earlier. However, they may be accessible to a broader range of learners using the functional reasoning presented here.

THE CANDLE PROBLEM

The desired time, when the slower-burning candle would be twice as long as the faster-burning candle, can be found using functional reasoning. Three solutions to this problem, each using a different function representation, follow.

Tabular Representation

We presented this problem to students in a collaborative problem-solving test setting and observed that their preferred method of solving it was by creating tables of the length of candle left at time t. **Figure 5** shows a sample response by one group of students. This group used proportional reasoning and made explicit reference to their reasoning with rates.

This group chose a candle of length 18 inches and used rate reasoning to create tables of time-length functions. Although these students did not use the word *function* in their discussion, they found a burn rate for both candles, expressed in inches per hour, and a burn rate for the faster-burning candle, expressed in inches per 30-minute interval. Students went on to make a nontrivial generalization of the problem by observing that the initial length was irrelevant. For this problem, the only factor that affects the time of interest is the burn rate because the initial lengths of the two candles were the same.

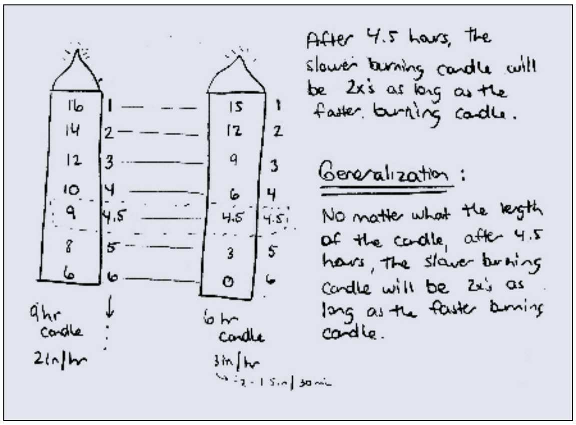

Fig. 5 Students used a table to solve the candle problem.

Algebraic Representation

Let $F(t) = 1 - (1/6)t$, $0 \le t \le 6$, and let $S(t) = 1 - (1/9)t$, $0 \le t \le 9$, be functions representing the fractions of the faster-burning and slower-burning candle, respectively, remaining at time t. Students could consider the following: How do these rates, expressed in terms of the fraction of the candle burned per hour, compare with the rates in inches per hour given by students whose work is shown in **figure 5**?

The desired ratio of the lengths of the two candles, 2:1, occurs when $S(t) = 2F(t)$. A solution to this equation is given below:

$$S(t) = 2F(t)$$
$$1 - \frac{1}{9}t = 2\left(1 - \frac{1}{6}t\right)$$
$$1 - \frac{1}{9}t = 2 - \frac{2}{6}t$$
$$\frac{2}{9}t = 1$$
$$t = 4.5 \text{ hr.}$$

Graphical Representation

By graphing $F(t)$ and $S(t)$, we can approximate when $S(t)$ is twice $F(t)$ (see **fig. 6**). Alternatively, because there appears to be a point (between 4 hours and 5 hours) at which $S(t) = 2F(t)$, it follows that if we graph $S(t)$ and $2F(t)$ on the same axes, then the graphs must intersect between 4 hours and 5 hours (see **fig. 7**).

From this graph, we can see that the desired time is $t = 4.5$ hours. To facilitate a discussion of the graphs, we ask the following types of questions:

- How many times is the length of the faster-burning candle half the length of the slower-burning candle? Explain the significance of your response in the problem context.
- What do the y-intercepts in **figure 6** and **figure 7** represent, if anything?

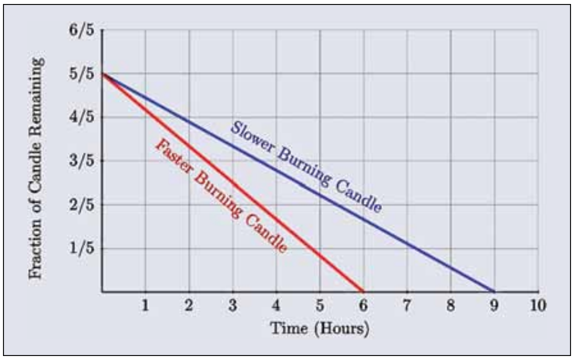

Fig. 6 The graphs show that somewhere between 4 hours and 5 hours $S(t)$ is twice $F(t)$.

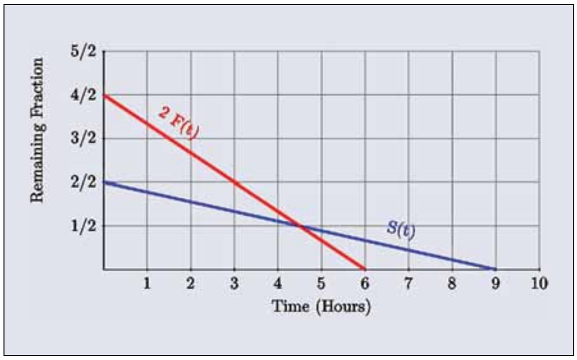

Fig. 7 Graphing $S(t)$ and $2F(t)$ provides an intersection point.

- Why are the y-intercepts of the two graphs the same in **figure 6** but different in **figure 7**?

Because most rate problems deal with combining the rates of two people or machines working together or against each other, this problem extends the rate problems discussion to include the comparison of two rates. Rarely do students have to focus on comparing rates of work and the effects of different rates of work. Note that the different representations used in this problem highlight different aspects of the functions. The graphical and tabular representations offer a generalized input-output process that makes it possible to estimate the actual solution by defining a small interval in which the solution must be contained.

Although the algebraic approach does not readily highlight such an interval, all three representations highlight the sameness of the y-intercepts for $S(t)$ and $F(t)$ and thus point to the fact that the rate of burning is the only factor to consider in determining the desired time.

Extensions of this problem may include starting with two candles of the same length or different lengths and changing the question as follows: For what rates is it possible for the faster-burning candle to burn to half the length of the slower-burning candle?

THE BIKER PROBLEM

Students' struggles with this problem demonstrate their misconception that the biker's average speed should be the mean of the average speeds of the two laps. Typically, students give an incorrect solution that is similar to the following:

$$\frac{25 \text{ mph} + x \text{ mph}}{2} = 40 \text{ mph}$$
$$25 \text{ mph} + x \text{ mph} = 80 \text{ mph}$$
$$x \text{ mph} = 55 \text{ mph}$$

The misconception here is that the biker would take the same amount of time to travel each of the two laps.

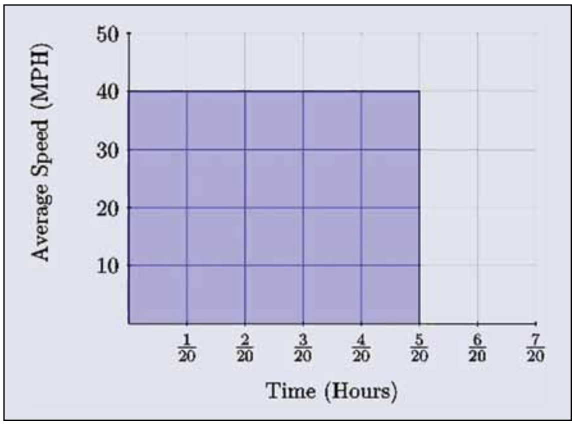

Fig. 8 The area of the rectangle represents the distance traveled.

We now present functional reasoning solutions, using both rate functions and distance functions. Consider **figure 8**, which shows a graph of the biker's average speed.

From our earlier discussion, we know that the area-under-the-rate graph represents work done. From this graph, we can see that the biker will need a total of 1/4 hour to complete the 10-mile distance. However, using

$$\text{time (hr.)} = \frac{\text{distance}}{\text{speed}} = \frac{5 \text{ mi.}}{25 \text{ mph}} = \frac{5 \text{ mi.}}{\frac{25 \text{ mi.}}{1 \text{ hr.}}}$$
$$= \frac{1}{5} \text{ hr.} = \frac{12}{60} \text{ hr.},$$

we know that he already used 12 minutes to complete the first lap, leaving only 3 minutes for the second lap. The question is reduced to this: What speed will allow the biker to cover 5 miles in 3 minutes? Alternatively, what speed will ensure that the area-under-the-rate graph in the interval from 12 minutes to 15 minutes, a 1/20-hour interval, is 5 miles? **Figure 9** shows the biker's speed over two laps; because the width of the area of interest is 1/20 hour, the height must be 100 mph.

Now let $R(t)$ represent the instantaneous rate of change function. Then

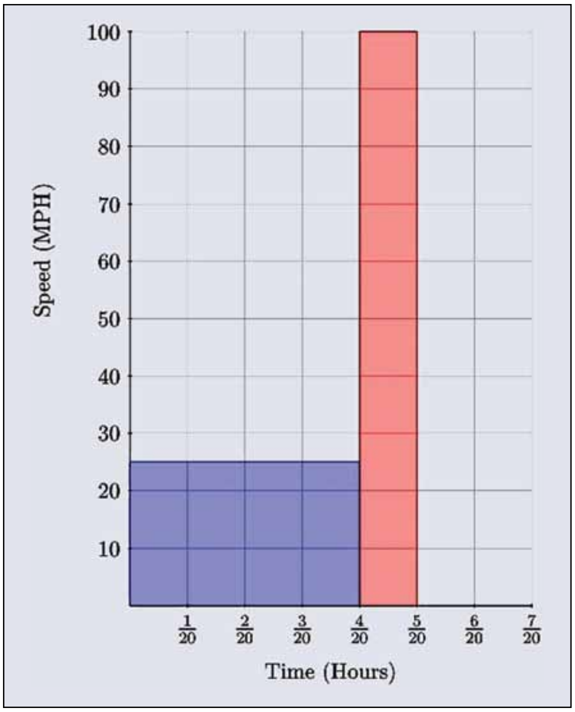

Fig. 9 The biker must average 100 mph on the second lap.

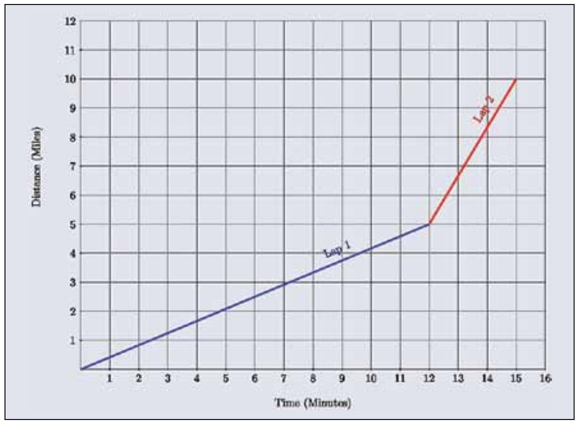

Fig. 10 The biker's distance traveled at time (t) is a piecewise function.

$$R(t) = \begin{cases} 25, & 0 \le t \le \frac{1}{5} \\ x, & \frac{1}{5} < t \le \frac{1}{4} \end{cases}$$

where x represents the biker's speed during the second lap. Using $R(t)$, we can find an x that gives an average speed of 40 mph:

$$\begin{aligned} \text{average speed} &= \frac{\text{total distance}}{\text{total time}} \\ 40 \text{ mph} &= \frac{10 \text{ mi.}}{\left(\frac{15}{60}\right) \text{hr.}} \\ &= \frac{(25 \text{ mph})\left(\frac{12}{60} \text{hr.}\right) + (x \text{ mph})\left(\frac{3}{60} \text{hr.}\right)}{\left(\frac{15}{60} \text{ hr.}\right)} \\ 10 \text{ mph} &= 5 \text{ mi.} + (x \text{ mph})\left(\frac{3}{60} \text{ hr.}\right) \\ 100 \text{ mph} &= x \text{ mph} \end{aligned}$$

Therefore,

$$R(t) = \begin{cases} 25, & 0 \le t \le \frac{1}{5} \\ 100, & \frac{1}{5} < t \le \frac{1}{4} \end{cases}.$$

The biker problem, as does the candle problem, stretches the discussion beyond what the typical problem might accomplish. The rate function is now a piecewise defined function, a concept students encounter and struggle with in precalculus. Students are engaged in a discussion of the distinction between average and instantaneous rates of change. Discussions about realistic speeds for a dirt bike as well as an instant change in speed from 25 mph to 100 mph are also relevant. Ask students whether, realistically, a biker who rides the first lap at 25 mph has any chance of qualifying.

Now consider approaching the problem using a distance-vs.-time graph (see **fig. 10**). For this graph, we use minutes as the units for time to emphasize that the time left for the second lap is much less than the time allowed for the first lap.

The goal is to find the total time needed to travel the two laps at the two speeds, which is the same as the time needed to travel the distance at a fixed speed of 40 mph. We already know that the biker traveled the first 5 miles at 25 mph; thus, we can expect that a line segment from $(t = 0, d = 0)$ to $(t = 12, d = 5)$ represents the distance-vs.-time graph during the first lap. Because the goal is to get to 10 miles before the time expires, it follows that the distance-vs.-time graph for the remaining lap is represented by a line segment connecting $(t = 12, d = 5)$ to $(t = 15, d = 10)$. The slope of the graph on that interval is 5 mi./3 min. Converting the slope to mph is a routine unit conversion:

$$\frac{5 \text{ mi.}}{3 \text{ min.}} = \frac{5 \text{ mi.}}{\frac{3}{60} \text{hr.}} = \frac{100 \text{ mi.}}{1 \text{ hr.}} = 100 \text{ mph}$$

Alternatively, let $D(t)$ represent the distance traveled by the biker at time t. Then

$$D(t) = \begin{cases} 25t, & 0 \le t \le \frac{1}{5} \\ 5 + xt, & \frac{1}{5} < t \le \frac{1}{4} \end{cases}$$

where x represents the biker's speed during the second lap. Using $D(t)$, we can find an x that gives a total distance of 10 miles:

$$10 \text{ mi.} = 5 \text{ mi.} + (x \text{ mph})\left(\left(\frac{1}{4}-\frac{1}{5}\right)\text{hr.}\right)$$

$$10 \text{ mi.} = 5 \text{ mi.} + (x \text{ mph})\left(\frac{1}{20}\text{hr.}\right)$$

$$100 \text{ mph} = x \text{ mph}$$

As with the rate graphs, this approach also involves piecewise defined functions and highlights a difference between average and instantaneous rate of change, two concepts that receive significant discussion in precalculus and calculus. Notice that the solution given using the distance function is equivalent to the following calculus question:

Solve

$$D(t) = 10 = \int_0^{\frac{1}{4}} R(t)dt$$

where $R(t)$ is the piecewise-defined rate function

$$R(t) = \begin{cases} 25, \ 0 \le t \le \frac{1}{5} \\ x, \ \frac{1}{5} < t \le \frac{1}{4} \end{cases}.$$

OPPORTUNITIES FOR TEACHERS

Functional reasoning is a central theme in the K–16 mathematics curriculum through which many concepts can be explored. The functional approaches presented here are avenues through which teachers can engage students in grade-appropriate activities to enhance and build on their functional reasoning, support conceptual understanding of rate, and prepare students for concepts that they will encounter later in the curriculum. If these functional approaches are appropriately implemented, concepts such as integrals, derivatives, and piecewise-defined functions will not be foreign to students when they reach calculus.

These approaches create opportunities for teachers to enrich discussions around rate problems and for students to make connections among important concepts such as the area of a rectangle, slope, and calculus concepts. In addition, functional reasoning lends itself to deeper and faster explorations of rate problems using technology. We offer these functional approaches not as replacements for more typical approaches but as alternatives to complement the traditional approaches while presenting opportunities for students to gain a better understanding of functions.

To expect students to discover the relationships discussed here without guidance is unrealistic. We help students make these connections through carefully planned questioning and deliberate use of multiple representations. Exploring multiple representations to represent rate and work functions, accompanied by questioning to scaffold the students' thinking about the functions, results in deeper understanding of these concepts.

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