Why Is the Midpoint an Average?

The relationship between a midpoint and an average showcases the interplay between procedural knowledge and conceptual knowledge in learning mathematics for teaching.

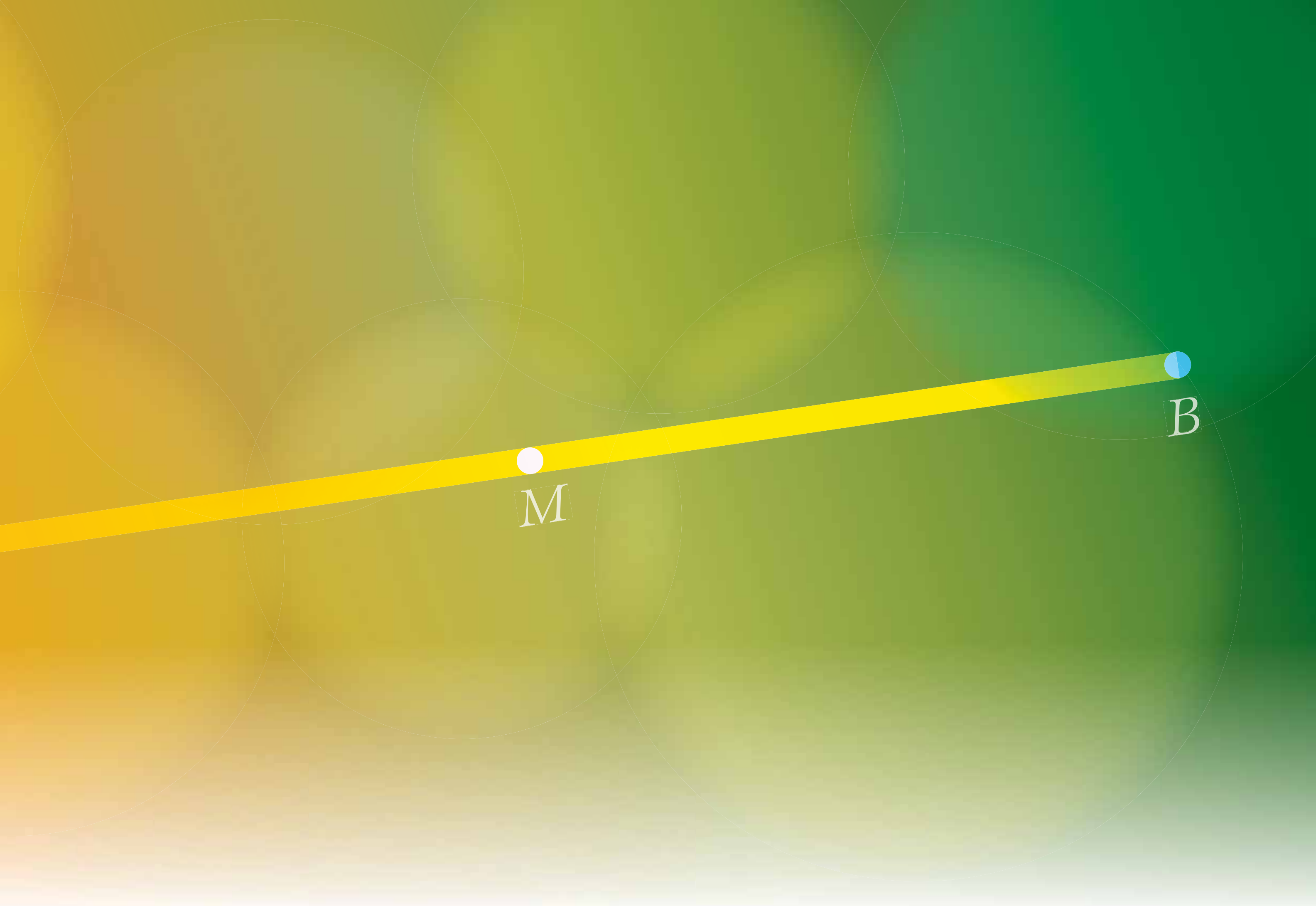
Lingguo Bu

Many preservice teachers come into teacher education programs with a rule-based view of mathematics, a view that is to a large extent indicative of their previous school experiences (Ball 1988; Ryan and Williams 2007; Roberts and Tayeh 2011). When asked to explore a new problem, they tend to respond, "I need a formula!" However, when a formula is given or known, such as the Pythagorean theorem, some may still have difficulty expressing the formula's meaning when they try to match its parts with those of the problem being modeled.

In general, preservice teachers seem to possess what Skemp (1978) characterized as an *instrumental understanding* of mathematics; although procedurally useful in everyday situations, it is strikingly distant from the kind of relational mathematical understanding that NCTM (1991, 2000) envisions for teachers and students. Research suggests that preservice teachers' preconceptions and misconceptions are valuable resources in reshaping and enriching their personal views about mathematics and mathematics teaching (Llinares and Krainer 2006; Ryan and Williams 2007).

In teaching preservice elementary school teachers, I have used their instrumental understanding as the starting point for rich discussions about the fundamental ideas of mathematics and meaningful mathematics teaching. Reshaping preservice teachers' mathematical understanding in methods courses is a challenging process. Preservice teachers must become convinced that their rules and formulas, which they have learned mostly by rote, have serious limitations. More important, these rules and formulas are grounded in solid mathematical relations. This is a unique aspect of preservice mathematics teacher education: Teacher educators must engage teacher candidates to unpack, or unlearn, their rule-based understanding of mathematics for the benefit of their future students and their own professional growth (Ball 1988).

In this article, I focus on preservice elementary school teachers' reconstruction of meaning for the midpoint formula, highlighting the interplay between instrumental and relational understanding (Skemp 1978)—in other words, the interplay between procedural and conceptual knowledge (Silver 1986) in learning mathematics for teaching. I further provide alternative perspectives on the relationship between midpoint and average to suit the needs of readers at various levels of mathematics learning and instruction.



THE CONTEXT

The discussion about midpoints was part of an elementary (K–9) content and methods course on Euclidean and coordinate geometries offered at Southern Illinois University Carbondale. At the time, two sections of the course were offered, and the combined enrollment was forty-five preservice teachers. Their initial responses in both sections were very similar regarding the midpoint of two points in the Cartesian system. Before the lesson on the midpoint formula, the preservice teachers reviewed the coordinates of points in the Cartesian system and developed the distance formula as a direct application of the Pythagorean theorem. During the midpoint lesson, they were expected to revisit and make sense of the midpoint concept and the formula through examples and geometric reasoning.

INITIAL REACTIONS TO THE MIDPOINT CONCEPT AND FORMULA

The preservice teachers were initially given a few pairs of special points and asked to find their midpoint. One problem stated, "Find the midpoint of the segment joining A (2, 5) and B (4, 7)." This was an easy exercise; it did not take long before someone observed that, given P (a, b) and Q (c, d), the coordinates of M, the midpoint of segment PQ, are $((a + c)/2, (b + d)/2)$. Most preservice teachers seemed to know the formula and were comfortable with the summary that the x-coordinate of M is the midpoint of the x-coordinates of P and Q and that the y-coordinate of M is the midpoint of the y-coordinates of P and Q (see **fig. 1**).

The student's observation above is based on a major assumption about the midpoint. So I asked the class, "Why is $(a + c)/2$ the midpoint of a and c if we are looking only at the x-coordinates of point P and point Q?" (see **fig. 1**). Some of the teachers referred to the examples, but the majority argued that $(a + c)/2$ was the *average* of a and c, as if the midpoint were the average of two points and vice versa and required no further explanation. Indeed, in many textbooks we can find similar statements, which typically are not followed by a detailed justification.

Could there be any deeper connection between the two concepts that has led to the fact that the numerical value of the midpoint and the average of the two numbers are the same? Or should we have just accepted this fact and moved on to more advanced ideas? The latter choice, obviously, was contrary to the course objectives, so I probed the teachers' thinking for a possible explanation.

THE INTERPLAY BETWEEN A FORMULA AND THE CONCEPT

When making arguments for the midpoint using the idea of an average (see **fig. 1**), the preservice teachers were apparently reinterpreting the formula rather than trying to find the underlying meaning of the term *midpoint*. They were familiar with the mathematical idea behind the average of two numbers—namely, a sum divided by 2—which does in fact correctly represent the coordinates of the midpoint. However, few of the teachers could provide an alternative or an explanation of the connection. It would be interesting to find out how students who have not previously seen the formula figure out the midpoint between two numbers, such as 6 and 14, on the number line.

The concept of midpoint is, literally, a geometric idea, whereas the concept of average is primarily an arithmetic or algebraic idea. A person with a relational understanding of the ideas is able to navigate through the multiple representations and the connections among them (NCTM 2000). However, the preservice teachers were focused mostly on the formula because of their learned habits about the use of mathematical formulas. Most formulas in mathematics are improved forms of a final product of lengthy reasoning; the product does not necessarily reveal the process. When finding the midpoint between a and c on the number line (assuming that $a < c$), we can readily reason that it is a plus half the difference between c and a. Therefore, the midpoint is

$$a + \frac{c-a}{2} = \frac{2a+c-a}{2} = \frac{a+c}{2}.$$

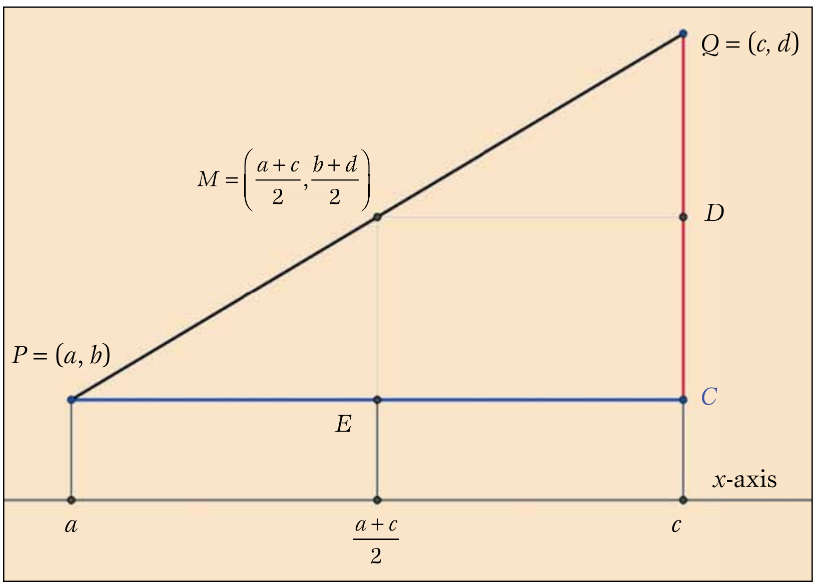


Fig. 1 The preservice teachers were aware of the rule for finding the midpoint of two points in the coordinate system.

Gelfand and Shen (1993) provide a similar justification in the context of the arithmetic mean of two numbers and their representations on the number line. It can be established in a meaningful manner that for any two points on the number line, the midpoint has the same value as the average of the two corresponding numbers. Although equating the midpoint with the average of two numbers (or points) on the number line may be mathematically harmless, this approach may pose challenges in other situations in which three or more numbers are involved and in which the average is certainly not the same as the midpoint.

A variety of similar cases in school mathematics demonstrate the rich connections among mathematical ideas. For example, a composite number is an integer that can be represented in a rectangular array of more than one row and column or, alternatively, as an integer that is the product of at least two nontrivial integer factors. The solution to a linear system in two variables, if unique, is the intersection of the graphs of the two equations or a pair of numbers that satisfy both equations. Indeed, the Pythagorean theorem and its converse are examples that integrate geometry and algebra in appealing ways. These mathematical connections are among the critical features of understanding in learning and teaching mathematics (NCTM 2000); thus, they should be carefully investigated with preservice teachers to reshape their views on the nature of mathematics and mathematics teaching.

In professional mathematical practices, formulas are the final products of mathematical reasoning and sense making. However, formulas do tend to redefine or offer alternative perspectives on the meaning of a mathematical concept, perspectives that may not be similar to the original line of thought.

The quadratic formula, for example, is originally a consequence of solving quadratic equations by completing the square. Once the formula is established, however, it allows us to analyze the solution set in terms of a discriminant. In the case of the midpoint, most preservice teachers are inclined to accept the formula as valid, requiring no further discussion. Nonetheless, when encouraged to explain, some preservice teachers could recreate a rationale for the mathematical idea on the basis of the formula. The formula, which may have been mastered by rote learning, now points to new dimensions of the original idea.

A classroom vignette relating to the midpoint formula demonstrates the complexity of preservice teachers' relearning or unlearning the mathematics that they have been exposed to in a largely instrumental manner.

JANE'S INTERPRETATION OF THE MIDPOINT FORMULA

Jane (all names are pseudonyms) was a nontraditional student in the class. She consistently tried hard to make connections and seek meaning about the class work. Regarding the midpoint of x_1 and x_2 on the number line, she accepted the average formula, as did the rest of the class. When asked to explain the formula, she made some drawings in her notebook (I had encouraged the teachers to make drawings) and, through a picture on a classroom tablet computer, shared her idea with the class (see **fig. 2**). Jane further provided an example to illustrate the validity of her explanation: The midpoint of 2 and 4 is $(2 + 4)/2$.

Jane's drawing caught the attention of Linda, who remarked, "It is just like folding the whole thing $[x_1 + x_2]$." In a numeric sense, Jane was still finding the average because she added the two pieces and divided the sum by 2. Further, she had trouble explaining why the average is the midpoint. Perhaps she had some intuitive ideas about the connections between the average—that is, the formula—and the midpoint. But Linda's comment led me to construct a plausible geometric explanation of the midpoint formula for the class.

On reflection, I realized that the complexity could have come from the multiple meanings of points on the number line. The term *point*, such as B, could be interpreted as a geometric point, a number, the distance from the origin O to the point B, or, alternatively, as the length of segment OB. Consequently, *midpoint* also has multiple meanings, depending on the context or a student's specific perspective. To find the midpoint of two points on the number line, we need to coordinate the multiple meanings of points.

In fact, Jane's idea can be extended to construct a geometric explanation for the midpoint formula, as shown in **figure 3**, where points A and B are interpreted as the distances from the origin O to A and B, respectively. To find the midpoint of A and B, we first connect $\overline{OA}$ and $\overline{OB}$ to make a long segment (shown as segment O_1A_1W in **fig. 3**). Then, we fold the whole segment O_1A_1W in half. In effect, we are folding the difference between OB and OA—that is, AB or A_1B_1—to get point M. Point M is therefore equally distant from A and B because of its construction. In other words, because segment A_1B_1 has the same buffer on both ends—the length of OA—folding the combined segment O_1A_1W in half splits the middle part, $\overline{A_1B_1}$, as well as the whole, yielding the desired midpoint M. Thus, in a geometric sense, M is the midpoint of A and B; in a numeric sense, it is the average of length OA and length OB, or half the sum of OA and OB.

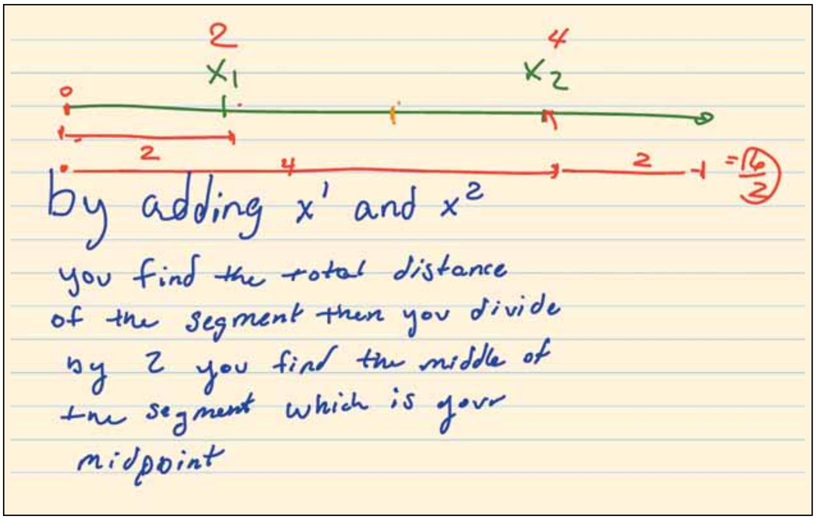


Fig. 2 Jane came up with an idea to explain the midpoint formula by connecting two segments (the superscripts shown here are not correct).

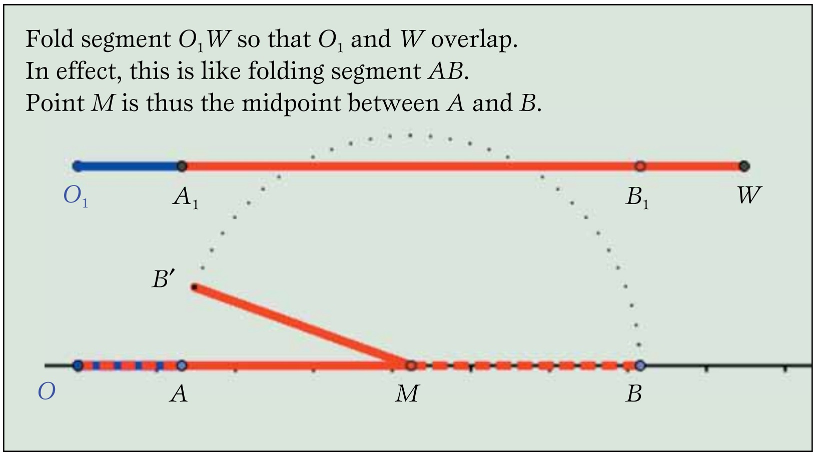


Fig. 3 The midpoint can be found by folding the combined segment in half.

ALTERNATIVE PERSPECTIVES

There is, in fact, another perspective on the midpoint-average connection: the idea of a convex combination, which characterizes the midpoint between points A and B on the number line as a special linear combination of A and B. A convex combination of points A and B is a point $C = s \cdot A + t \cdot B$, with $s, t \geq 0$ and $s + t = 1$. Geometrically, C can be any point along segment AB. Similarly, a convex combination of three points A, B, and C in the plane is a point D such that $D = s \cdot A + t \cdot B + u \cdot C$, with $s, t, u \geq 0$ and $s + t + u = 1$. Geometrically, point D can be any point on $\triangle ABC$, including its interior.

Let a, b, m be the coordinates of, respectively, A, B, and their midpoint M on the number line. Then, $m = s \cdot a + t \cdot b$, with $s, t = 1/2$. Although convex combinations are less relevant in school mathematics, they can be used as a pedagogical prompt to design an informal, dynamic model to illustrate the connection between the ideas of a midpoint and an average.

Because, in a metaphorical sense, an average has the connotation of give-and-take to reach the

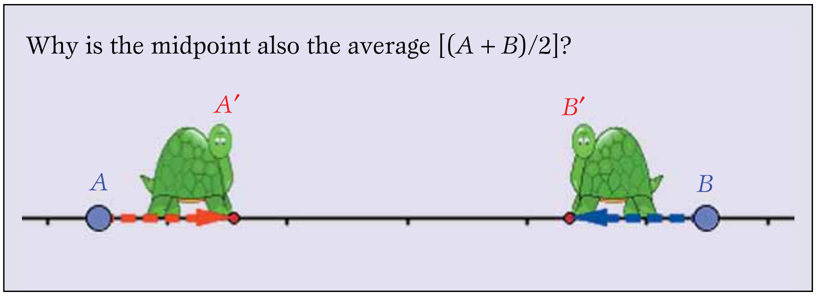


Fig. 4 A dynamic model shows the connection between the midpoint and the average metaphorically.

same level, we can design an activity as shown in **figure 4**. Given points A and B, we relate them to two turtles (or any moving objects), A' and B', respectively. Initially, A' is at A and B' at B. Then, both turtles start moving at the same speed in very small steps toward each other. Whereas B' *gives* away a certain advantage, A' *takes* that advantage. The two points are therefore moving toward a common midpoint, and, accordingly, the idea of an average is realized as the midpoint, because both have traveled the same distance when they meet in the middle. Of course, we assume that the turtles would walk in such a way that they would not walk past each other. As a metaphor, this dynamic model has its advantages. In light of the balance of give-and-take, it is straightforward to show that $(a - m) + (b - m) = 0$, a property of average that can be extended to multiple points on the number line.

If we move one step further, we could develop an algebraic explanation. Because a midpoint M is a point that is equally distant from two endpoints (assuming that $a < b$), then $m - a = b - m$, which gives $m = (a + b)/2$ (e.g., Gelfand and Shen 1993).

Extension to Midpoints in the Plane

The folding perspective on the midpoint can be extended from the number line to the plane to explain the formula for the midpoint of points P and Q (refer to **fig. 1**). Because two coordinates are related to a point in the plane, finding the midpoint directly by folding segment PQ is difficult. However, we could find the horizontal midpoint E of the x-coordinates and the vertical midpoint D of the y-coordinates of point P and point Q, respectively (see **fig. 1**). Then we need to show that E and D correspond to the x- and y-coordinate, respectively, of the midpoint of $\overline{PQ}$. For that goal, we draw a vertical line through E that is parallel to the y-axis and intersects $\overline{PQ}$ at M. Next, we connect M and D. We need to show that $\overline{MD}$ is parallel to $\overline{EC}$. We reason as follows.

Because $\triangle PEM$ and $\triangle PCQ$ are similar and E is the midpoint of $\overline{PC}$, then M is the midpoint of $\overline{PQ}$ and $\overline{ME}$ is congruent to $\overline{DC}$. It is clear that quadrilateral $CDME$ is a rectangle and therefore $\overline{MD}$ is parallel to $\overline{EC}$. Thus, the x-coordinate of E and the y-coordinate of D are, respectively, the x- and y-coordinates of M.

EXTENDING THE VIGNETTE

Many ideas in elementary school mathematics are sophisticated in their multidimensional connections and thus have profound implications for teaching and learning mathematics with understanding (Ma 1999; Harel and Sowder 2005; Wu 2009).

The midpoint vignette is just one case, with its obvious limitations for knowledge transfer, but it has a place in the education case knowledge for mathematics teachers for its potential to guide "the work of a teacher, both as a source for specific ideas and as a heuristic to stimulate new thinking" (Shulman 1986, p. 12) about mathematics and mathematics teachers. At another level, the midpoint vignette has been enlightening for the author himself as he strives to understand the complexity of educating mathematics teachers. Whether the midpoint scenario could be replicated with other groups of preservice teachers under a teacher educator's guidance remains an open question. Further, numerous cases in preservice teacher education share similar themes—such as dividing fractions, setting up basic proportions, multiplying negative numbers, or solving linear equations—for which the midpoint vignette may be informative.

So why, from the perspective of educating mathematics teachers, is the midpoint an average? It is a mathematical fact that can be established through sense making and reasoning across geometry, arithmetic, and algebra, including teacher candidates' personal interpretations. A relational understanding of the midpoint and the corresponding average consists in making meaningful connections across the multiple dimensions of the mathematical idea. In essence, "understanding involves making connections" (NCTM 2000, p. 64).

ACKNOWLEDGMENTS

The author wishes to thank Wesley Calvert of the Department of Mathematics at Southern Illinois University Carbondale, who referred him to convex combinations and the idea of dynamic modeling.

REFERENCES

Ball, Deborah L. 1988. "Unlearning to Teach Mathematics." *For the Learning of Mathematics* 8, no. 1: 40–48.

Gelfand, Israel M., and Alexander Shen. 1993. *Algebra*. Boston: Birkhäuser.

Harel, Guershon, and Larry Sowder. 2005. "Advanced Mathematical Thinking at Any Age: Its Nature and Its Development." *Mathematical Thinking and Learning* 7, no. 1: 27–50.

Llinares, Salvador, and Konrad Krainer. 2006. "Mathematics (Student) Teachers and Teacher Educators as Learners." In *Handbook of Research on the Psychology of Mathematics Education: Past, Present, and Future*, edited by Angel Gutiérrez and Paolo Boero, pp. 429–59. Rotterdam: Sense Publishers.

Ma, Liping. 1999. *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics (NCTM). 1991. *Professional Standards for Teaching Mathematics*. Reston, VA: NCTM.

———. *Principles and Standards for School Mathematics*. 2000. Reston, VA: NCTM.

Roberts, Sally K., and Carla Tayeh. 2011. "All These Rays! What's the Point?" *Mathematics Teaching in the Middle School* 16, no. 7: 408–13.

Ryan, Julie, and Julian Williams. 2007. *Children's Mathematics 4–15: Learning from Errors and Misconceptions*. New York: Open University Press.

Shulman, Lee S. 1986. "Those Who Understand: Knowledge Growth in Teaching." *Educational Researcher* 15, no. 2: 4–14.

Silver, Edward A. 1986. "Using Conceptual and Procedural Knowledge: A Focus on Relationships." In *Conceptual and Procedural Knowledge: The Case of Mathematics*, edited by James Hiebert, pp. 181–98. Hillsdale, NJ: Lawrence Erlbaum Associates.

Skemp, Richard R. 1978. "Relational Understanding and Instrumental Understanding." *Arithmetic Teacher* 26, no. 3: 9–15.

Wu, Hung-Hsi. 2009. "What's Sophisticated about Elementary Mathematics?" *American Educator* 33, no. 3: 4–14.

LINGGUO BU, lgbu@siu.edu, is assistant professor of mathematics education at Southern Illinois University in Carbondale. His interests include the integration of mathematical modeling and dynamic technologies in teacher education and professional development.