GEOMETRIC REASONING ABOUT A CIRCLE PROBLEM

An open-ended problem about a circle illustrates how problem-based instruction can enable students to develop reasoning and sense-making skills.

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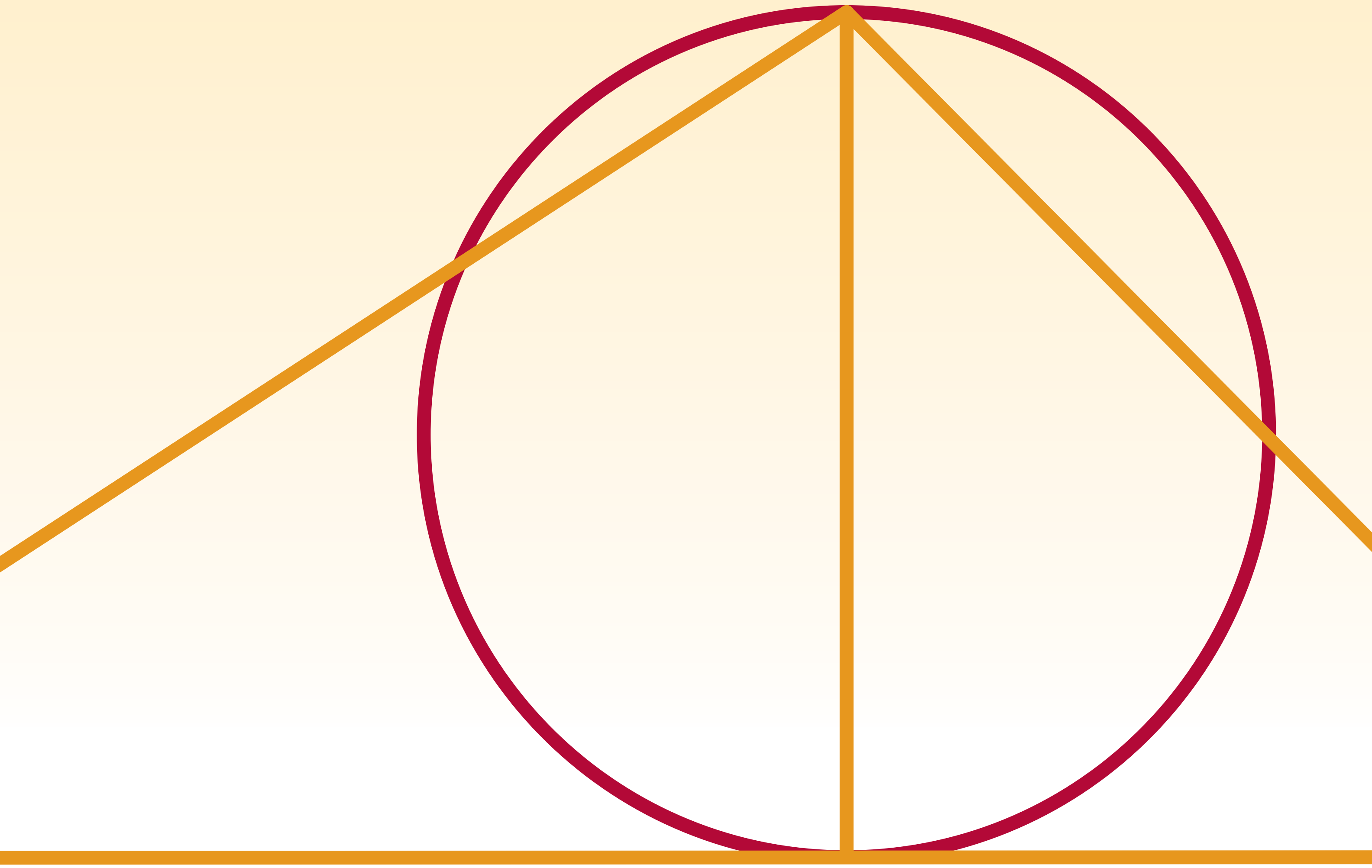
What does problem-based instruction do for students and teachers? The open-ended geometry problem presented here along with examples of students' work on the problem illustrates how problem-based instruction can help students develop their mathematical proficiency.

NCTM's *Principles and Standards for School Mathematics* (2000) has promoted the use of problem-based instruction for teaching mathematics with understanding (p. 52). Problem-based instruction provides opportunities for students to develop their reasoning and sense-making skills (NCTM 2009). Recent studies have shown that students who experience problem-based instruction improve their perceptions about mathematics and about themselves as mathematical learners (Ridlon 2009). Moreover, instruction that incorporates problem-based lessons does not compromise students' development of the mathematical skills that are promoted when using traditional curricula. At the same time, problem-based instruction pushes students to use multiple problem-solving strategies



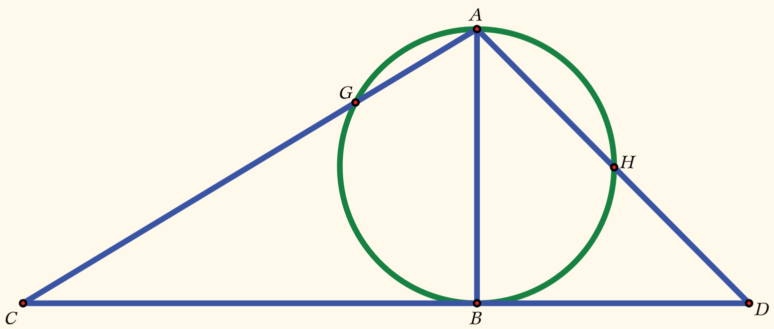
and to work on more cognitively demanding tasks (Cai et al. 2010; Ridgway et al. 2003). However, some teachers still have difficulty understanding how to make problem-based instruction a reality in their classrooms.

In this article, we show an example of a problem-based lesson taught in a geometry classroom—the circle problem (see **fig. 1**)—and provide evidence of geometry students' reasoning and sense making when working on it. The circle problem provided a context for students to develop competency in the five strands of mathematical proficiency outlined in *Adding It Up*: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick, Swafford, and Findell 2001).

Our analysis of students' work focuses on the latter three competencies, which are less commonly discussed in the literature. *Strategic competence* refers to the ability to formulate, represent, and solve mathematical problems; *adaptive reasoning* is the capacity for logical thought, reflection, explanation, and justification; and *productive disposition* refers to the inclination to see mathematics as worthwhile combined with a belief in diligence and one's own efficacy (Kilpatrick, Swafford, and Findell 2001, p. 116). By developing insight into how students can build mathematical proficiency through specific problem-based lessons, teachers may become more equipped to teach mathematics through problems.

The circle problem was used in two sections of a regular high school geometry class in a large urban midwestern school. The first author, in collaboration with a teacher whom we call Mr. Brown, developed a lesson plan with the circle problem (see **fig. 2**), which is based on an exercise in a traditional geometry textbook (Moise and Downs 1975, p. 488). The variation of the problem was innovative because students could perceive the problem as one of numerical calculation; on inspection, however, much more than performing calculations was required. Moreover, the problem anticipated a theorem that students would study soon, thus instilling in them the need to prove that theorem.

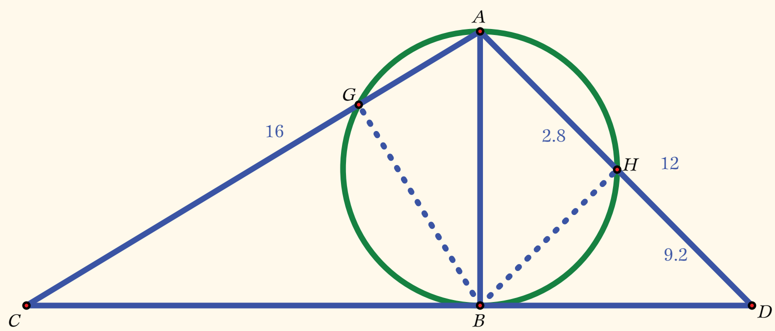
The problem was given at the end of the year, when students were studying a unit on circles. At that point, the class had already covered the mathematical concepts relevant for solving the problem—for example, the Pythagorean theorem, similar triangles, theorems about the geometric mean in right triangles, and trigonometry.



AB is a diameter. CD is tangent to the circle at B. $AC = 16$ cm, $AD = 12$ cm, and $AH = 2.8$ cm. Find as much information as possible about the segments and the angles in the diagram.

Fig. 1 Students worked in groups to solve the circle problem.

In the diagram below, $\overline{AB}$ and $\overline{CD}$ are perpendicular. Adding the auxiliary lines BH and BG, we notice that angles AHB and AGB intercept the diameter; so apply the theorem about inscribed angles to conclude that angles AHB, DHB, AGB, and CGB are all right angles.



Notice that BH is the geometric mean of AH and HD. Therefore, we can set up the ratio

$$\frac{2.8}{BH} = \frac{BH}{9.2}$$

to find that $BH \approx 5.08$, where we round to the nearest hundredth. Now, use the Pythagorean theorem in triangles BHD and ABD to find that $BD \approx 10.51$ and $AB \approx 5.79$. Knowing AB, we can compute that $CB \approx 14.92$. Now we can consider triangle ABC and notice that BC is the geometric mean of CG and CA. So we set up the ratio

$$\frac{x}{14.92} = \frac{14.92}{16}$$

and find that $CG \approx 13.91$. Therefore, $GA \approx 2.09$.

Fig. 2 One solution involved using the geometric mean.

The geometric mean occurs in two theorems about right triangles. One theorem states that the measure of the altitude to the hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse; the other states that the measure of a leg is the geometric mean between the measures of the hypotenuse and the segment of the hypotenuse adjacent to that leg.

Students had not yet learned the secant-tangent product theorem, so the problem served as an introduction to that theorem and the techniques necessary for proving it. Thus, the circle problem gave students the opportunity to review theorems that they had previously studied and to apply those theorems in a new context. Because the circle problem was designed to allow students to develop their own strategies, it did not name any theorems explicitly, and students had to recall relevant knowledge to solve the problem.

Mr. Brown allowed students to work on the problem for two days. During day 1, the problem was introduced, and then students worked in small groups of three or four for the entire class period. In his introduction, Mr. Brown mentioned theorems about circles that students had studied recently but did not mention concepts that they had studied earlier in the year and that were fundamental for solving the problem, such as similarity, the Pythagorean theorem, or the geometric mean. During day 2, Mr. Brown led a whole-class discussion of the solution of the problem. Our analysis is based on videos from day 1, while students were working on the problem in small groups, and worksheets from students in one class. From these we show how the problem enabled students to build strategic competence, adaptive reasoning, and a productive disposition.

BUILDING STRATEGIC COMPETENCE

One benefit of problem-based mathematics instruction is that it allows students to decide what the problem is and the best way to solve it. With the circle problem, Mr. Brown did not give students specific guidelines about how to solve it. Instead, he told them to find as much information about the diagram as possible, and they had to decide on their own what strategies they should apply to solving the problem.

As a result, students applied various strategies. In our analysis of twenty-two students' worksheets from one class, we found that nearly all students used multiple strategies to work on the problem, relying on key features of the diagram. **Table 1** lists all the strategies that students used and the number of students who demonstrated evidence of using each strategy. The twenty-two students whose work we analyzed used a total of sixty strategies; thus, on average, each student used about three different strategies to work on the problem.

Students' use of the various strategies for working with the diagram, such as adding auxiliary lines and marking right angles on the diagram, is evidence of their strategic competency because the diagram provided did not include everything that they needed to solve the problem. For example, the problem said that segment *AB* was a diameter and that segment *CD* was a tangent. However, the given diagram did not include markings for denoting that $\overline{AB}$ and $\overline{CD}$ were perpendicular. Therefore, students would have to identify relevant information in the problem and add the relevant markings to the diagram.

By providing opportunities for students to add markings to the diagram, Mr. Brown made students accountable for recalling prior knowledge necessary for solving the problem and allowed them to integrate information about a geometric figure into the diagram. Because students were expected to draw auxiliary lines (without a cue that such lines were needed), they had to reason about strategies that could be helpful for solving the problem by transforming the diagram and also about the justification for adding such lines.

Table 1 Strategies for Working with the Diagram

Strategy	Number of Students Who Displayed Use of This Strategy
Adding auxiliary lines	21
Marking right angles on the diagram	21
Redrawing parts of the diagram	8
Changing the orientation of parts of the diagram	8
Marking congruent pieces on the diagram	2

DEVELOPING ADAPTIVE REASONING

The circle problem offered multiple starting points and solution strategies for students. From our analysis of students' worksheets, we learned that students drew on their prior knowledge about different geometric concepts to solve the problem. One widely-used strategy was to apply the Pythagorean theorem to find the side lengths of right triangles. Identifying and applying similar triangles and the geometric mean was a less common approach, although using at least one of these techniques was essential to solve the problem completely.

Figure 3 shows a summary of the prior knowledge that students drew on, based on our examination of the twenty-two worksheets. For example, six students used the Pythagorean theorem and similar triangles; three students used both those concepts as well as the geometric mean; and two students used only the Pythagorean theorem. In their use of multiple strategies to solve the problem, students showed their capacity for logical thought about how certain elements of the problem were related and also their ability to justify their computations.

Students' choice of using similar triangles or the geometric mean to solve the problem is an example of adaptive reasoning. They applied their knowledge about specific theorems and properties of geometric objects to the specifics of the circle problem. After drawing auxiliary segment *BH* in the diagram, students could find *BH* either by identifying similar triangles or by applying the geometric mean.

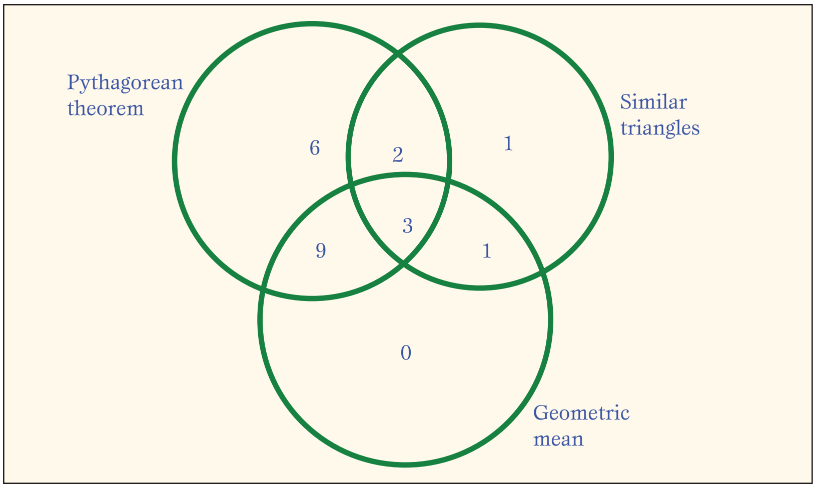


Fig. 3 Students used some or all of these concepts in their solutions.

Research on learning suggests that learners' ability to organize their prior knowledge so that they can make adaptations to new situations is crucial for learning (NRC 2000, pp. 45–48). Therefore, a problem that enables students, on their own, to make connections with concepts that they have studied before could aid in the development of adaptive reasoning. Students in Mr. Brown's class had studied similar triangles and the geometric mean earlier in the semester. Their capacity to remember these concepts on their own and, moreover, to justify why they were relevant to the solution of the problem were crucial for developing adaptive reasoning.

FOSTERING A PRODUCTIVE DISPOSITION

To develop strategic competence and the ability to reason adaptively, students must see mathematics as a worthwhile pursuit and develop habits of mind to apply consistent and diligent effort (Kilpatrick, Swafford, and Findell 2001). A key element of a successful problem-based lesson is the chance for students not only to make sense of the mathematics involved but also to understand the benefits of persevering when the solution is not immediately apparent. In our analysis of the classroom videos, we found evidence of the challenges that students encountered as well as their efforts to overcome those challenges.

One group of three students initially had some difficulty drawing auxiliary lines that would be helpful for finding the length of diameter AB. After looking in their textbook for a relevant theorem, they first found the theorem that states that a line parallel to one side of a triangle divides the other two sides proportionally. Given this information, they chose to construct a line segment from H parallel to $\overline{AB}$ (see **fig. 4**).

An example of students' dialogue as they worked on the problem follows:

Student 1: OK, if a line parallel to the side of the triangle intersects the other two sides, then it divides those two sides proportionally, which means … But wait, that still doesn't work.
Student 2: Why?
Student 1: Because we don't have a line. Because we don't know what this length is [*pointing to the auxiliary line*]. Because, OK, because then you could say that, um, if they're parallel, then 9.82 or, like, 2.8 over 9.2 equals the diameter over that length. But we don't know what that length is, so then that doesn't work.

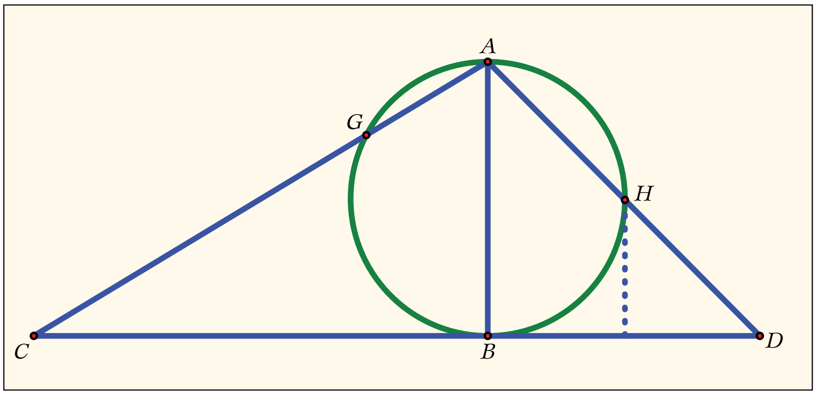


Fig. 4 A first attempt at choosing an auxiliary line was unproductive.

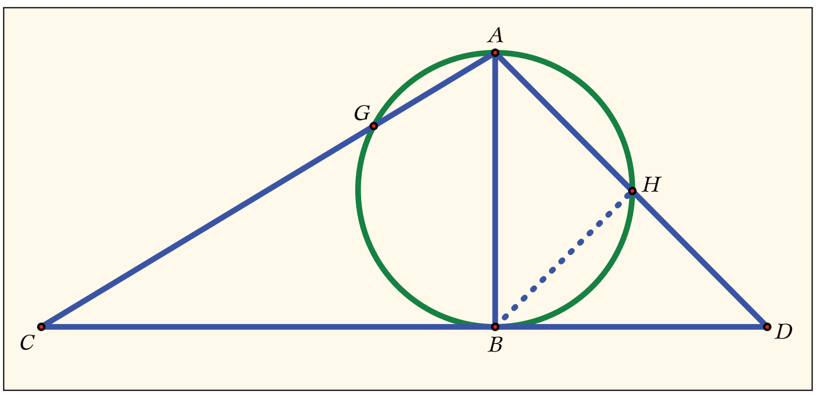


Fig. 5 The group eventually erased its first line and drew $\overline{BH}$.

These students realized that, although they had found a way to draw an auxiliary line and apply the theorem about a line parallel to a side of a triangle, they did not have enough information to make the theorem useful. They erased the line they had drawn and continued to work, eventually arriving at a more successful strategy (see **fig. 5**).

Student 1: Oh, OK, so we can find this right here [*draws* $\overline{BH}$] is equal to this times this [*pointing to* $\overline{AH}$ *and* $\overline{HD}$]. This squared is equal to this times this. So then we can figure out this length [*pointing to* $\overline{BH}$].

These students then used the geometric mean to find BH and applied the Pythagorean theorem to find AB. Although their first attempt did not provide the solution to the problem, it helped them eventually arrive at the correct solution. The students' discussion took into account the need to establish a relationship between $\overline{AB}$ and some auxiliary line, but they learned that certain relationships did not lead to the solution of the problem. Eventually, when they found the relevant theorem and drew the appropriate auxiliary line, they were able to use the information that they had to solve the problem. The students in this group showed a productive disposition because they persisted, learning from previous attempts. Their productive disposition manifested itself in the students' ability to learn from their mistakes by looking for new relationships, studying the diagram carefully, and making connections with theorems that they already knew.

The dialogue excerpts above show that students developed greater mathematical proficiency through the complementary relationship between the different strands. The students showed strategic competence by solving the circle problem. In addition, they showed adaptive reasoning by evaluating different solutions and justifying their solution methods. Their success in solving the problem gave them the opportunity to develop their productive disposition further because they could see how their mathematical efforts paid off.

REASONING AND SENSE MAKING WHEN WORKING ON THE CIRCLE PROBLEM

The circle problem helped Mr. Brown promote reasoning and sense making in his mathematics classroom. We use the four key elements of reasoning and sense making set forth in *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009, pp. 9–10) to illustrate how these played out in the problem:

- *Analyzing the problem*—In approaching the problem, students first had to analyze the diagram and find the hidden structure by drawing auxiliary lines.
- *Implementing a strategy*—Students decided on a strategy for finding the length of the diameter and kept track of their computations.
- *Seeking and using connections across different mathematical domains …* —Students made connections between different mathematical concepts. For example, to apply the Pythagorean theorem to certain triangles, they had to first use the geometric mean or similar triangles to find side lengths.
- *Reflecting on a solution to a problem*—When students implemented an incorrect strategy, this time of reflection allowed them to realize their error and move on to a new strategy. Once they had chosen a more productive strategy, they were able to carry out their computations.

Reasoning and sense making are essential to the development of mathematical proficiency (Kilpatrick, Swafford, and Findell 2001). As a result of our observations of students' work, we propose that the following characteristics of the problem-based lesson built around the circle problem could have supported the reasoning habits that we have discussed and thus allowed students to become more mathematically proficient.

The lesson—

- allowed students to articulate the problem and their strategies for solving it, such as adding auxiliary lines;
- required students to think logically about connections between theorems regarding circles that they had studied recently and theorems about triangles that they had studied in the past;
- required students to take different pieces of knowledge and organize them into a coherent whole—for example, the knowledge about segment lengths that had been provided, the knowledge about characteristics of the diagram that they had to identify, and the knowledge of the theorems and strategies that they could use to solve the problem;
- encouraged students to use procedures efficiently and accurately for identifying unknowns, setting up proportions, and solving equations; and
- gave students the chance to experience the benefits of perseverance by trying different strategies until these strategies were fruitful.

These lesson components can help teachers promote reasoning and sense making in the mathematics class as students work through problems.

Reasoning and sense making are essential to developing the strands of mathematical proficiency. The circle problem required students to reason mathematically, thus enabling them to improve their reasoning as well as their strategic competence and productive disposition. If the goal of instruction is to help students develop mathematical proficiencies, then we should provide examples like the circle problem to give students a chance to develop their ability to reason mathematically.

REFERENCES

Cai, Jinfa, Ning Wang, Bikai Nie, John C. Moyer, and Chuang Wang. 2010. "Student Learning and Classroom Instruction Using Standards-based and Traditional Curricula: An Analysis of Instructional Tasks." Paper presented at the annual meeting of the American Educational Research Association, Denver.

Kilpatrick, Jeremy, Jane Swafford, and Bradford Findell, eds. 2001. *Adding It Up: Helping Children Learn Mathematics*. Washington, DC: National Academy Press.

Moise, Edwin E., and Floyd L. Downs. 1975. *Geometry*. Reading, MA: Addison-Wesley.

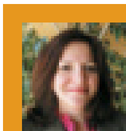
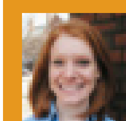
National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

———. 2009. *Focus in High School Mathematics: Reasoning and Sense Making*. Reston, VA: NCTM.

National Research Council (NRC). 2000. *How People Learn: Brain, Mind, Experience, and School.* Washington, DC: National Academy Press.

Ridgway, James, Judith Zawojewski, Mark Hoover, and Diana Lambdin. 2003. "Student Attainment in the Connected Mathematics Curriculum." In *Standards-Based School Mathematics Curricula: What Are They? What Do Students Learn*? edited by Sharon Senk and Denisse Thompson. Mahwah, NJ: Lawrence Erlbaum.

Ridlon, Candice L. 2009. "Learning Mathematics via a Problem-Centered Approach: A Two-Year Study." *Mathematical Thinking and Learning* 11: 188–225.

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