COUNTER
AS STARTING POINTS FOR REASONING

Asked to "fix" a false conjecture, students combine their reasoning and observations about absolute value inequalities, signed numbers, and distance to write true mathematical statements.

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How do we generate a culture of reasoning and sense making in our classrooms? One way is to encourage students to investigate mathematical statements even after they have found them to be false. Demonstrating that a mathematical generalization is false requires only one counterexample, but sense making demands additional steps.

Such practice is well grounded in the work of mathematicians, whose quest for sense making is continuous. It is not unusual for a mathematician to write a *mathematical proposition* (a statement that he or she believes to be true), only to discover that it is false. A mathematician might then ask whether the proposition can be altered to create a mathematical truth. Doing so while salvaging as much of the original conjecture as possible often involves informal observations and reasoning—one way in which mathematicians discover new mathematics.

Given the call for reasoning and proving to play an important role in school mathematics (NCTM 2000, 2009), as well as growing interest in school mathematics materials that better align with the parent field (RAND 2003), mathematics teachers need articles that document efforts to create classroom experiences that address these lofty goals. This article describes a recent classroom activity with college sophomores in a methods-of-proof course in which students reasoned about absolute value inequalities. The course was designed to meet the needs of both mathematics majors and secondary school mathematics teaching majors early in their college studies.

Addressed in particular is the challenge of navigating the interplay between formal reasoning (e.g., proving) and informal reasoning (e.g., observations from examples) to generate a culture of sense making. NCTM's *Focus in High School Mathematics: Reasoning and Sense Making* (2009) tells us that "mathematical reasoning can take many forms, ranging from informal exploration and justification to formal deduction, as well as inductive observations" (p. 4) and that "reasoning and sense making are intertwined across the continuum from informal observations to formal deductions" (p. 4). This article offers another dimension to this perspective by demonstrating how students can move from formal to informal reasoning and back again and how all types of reasoning can work harmoniously and nonlinearly toward an ultimate objective of making sense of the mathematics being studied.

The topic and mathematics content are appropriate for high school mathematics courses (see Ellis and Bryson [2011] for a discussion of similar content in high school courses). Further, the instructional strategy of asking students to reason about

EXAMPLES AND SENSE MAKING

the examples they provide is relevant for any high school or college mathematics course.

THE TASK

During a unit on inequalities, students were asked to explore the following conjecture:

$$|x - y| \le |x| - |y|$$

The term *conjecture* was used in this course to describe mathematical sentences without any connotation of truth values. The term *proposition* was used to describe statements that someone asserts to be true, and *theorem* was reserved for statements proven to be true. This terminology allowed me to pose questions about mathematical statements to students without giving any hint of whether or not the statements were true.

The quantifiers were purposely left off the conjecture because they are frequently omitted in mathematics textbooks. The students were accustomed to identifying when statements are open statements (such as equations to be solved) and when statements are quantified with an implicit "for all," asserting that something is always true. This particular conjecture was the latter, a for-all statement that can be rewritten more formally as follows:

> For all real numbers x and y, $|x - y| \le |x| - |y|$. Equivalently, if x and y are real numbers, then $|x - y| \le |x| - |y|$.

Working individually, students quickly found counterexamples and recorded them. **Figures 1** and **2** are two examples of student work.

Sam correctly asserts that the conjecture is false and provides a suitable counterexample (see **fig. 1**). She also carefully identifies the important formal aspects in her work:

1. She rewrites the conjecture using the appropriate quantifier in the form "for all x, y."
2. She points out that proving the negation of the conjecture (which she writes correctly as a "there exists" statement) demonstrates that the conjecture is false.

Pat also gives an excellent response (see **fig. 2**). He writes the conjecture in proper notation (using the symbol $\forall$ to denote "for all"), and he writes his counterexample in proper form, noting that his

Conjecture 11: $|x-y| \le |x| - |y|$

For all $(x \text{ and } y) \in \mathbb{R}$ $|x-y| \le |x|-|y|$

negation: There exist an $(x \text{ and } y) \in \mathbb{R}$ such that $|x-y| > |x|-|y|$

Example: $x=0$, $y=-1$

so $|x-y| = |0-(-1)| = |1| = 1$

and $|x|-|y| = |0|-|-1| = -|-1| = -1$

and $1 > -1$ $[|x-y|>|x|-|y|]$

The negation is true so the conjecture is false.

Fig. 1 If the negation of the statement is true, the statement is false.

choices of values for x and y satisfy the hypothesis, not the conclusion.

Although I was pleased with the students' work, I felt that we could learn more. Even though we had answered the question of whether or not this conjecture is true, as mathematical thinkers we could go further. *Making sense* can be described as understanding the situation, context, or concept by connecting it with existing knowledge (NCTM 2009). Here the context was comparing inequalities that connect to concepts of distance and signed numbers. Conjecturing about how to "fix" the original statement would result in making sense of the original conjecture. By "fix," I meant adjusting the hypothesis or the conclusion so that a theorem (a true proposition) is formed while salvaging as much of the original conjecture's intent as possible.

$\forall x,y \in \mathbb{R},\ |x-y| \le |x|-|y|$

False

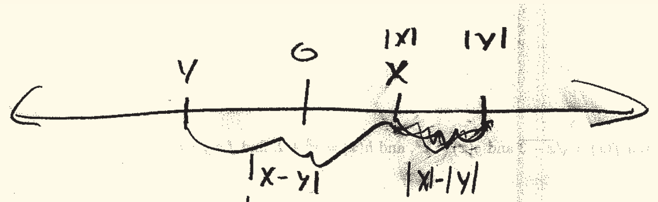
Pf/ Let $x=0,\ y=1$

$0,1 \in \mathbb{R}$ and $|-1|=1 \not\le -1 = |0|-|1|$

Fig. 2 Producing a counterexample also shows that the statement is false.

Discussion: I first thought it could be false because when the entire side has absolute values it must be a positive number ($|x-y|>0$) but since the entire other side isn't and there is a negative sign outside of the absolute value signs, it could be a negative number ($|x|-|y|$ could be <0) and if it is it is less than the positive number on the other side which s a contradiction to it being greater than or equal to.

(a)



distance between x and y

$|x-y|$ distance between

$|x|-|y|$ differance of distances from zero.

(b)

Fig. 3 Student responses tended to fall into the two categories, as the examples here indicate.

This process is obviously subjective because "the original intent" is in the eye of the beholder.

I could have simply asked students to write down a "fixed" statement, but I also wanted them to understand that informal thinking is important in mathematics. Moreover, I wanted them to connect their fix to existing knowledge, which now included observations and counterexamples. In particular, I wanted them to note that although empirical evidence will not prove the modified conjecture (assuming that they wrote a general statement), empirical evidence (i.e., the counterexamples) could be valuable in conjecturing about possible fixes.

I gave students the following directions:

1. Take a moment to reflect on how you found your counterexample. For example, was it a guess-and-check situation, or did you make some sort of observation that led you to a counterexample?
2. Think about what your counterexample tells you about why this statement is true. Summarize your thoughts in a short paragraph.

THE NEXT STEPS

Over the next two evenings, I sorted the students' responses. Student observations that were clearly expressed fell into two categories; a few other responses expressed ideas that were difficult to classify. One type, shown in **figure 3a**, mentioned that one side of the inequality could be negative whereas the other is always positive. Ten responses fit into this category. The other type of clearly articulated response demonstrated that the distance between x and y could be greater than the distance between $|x|$ and $|y|$ (see **fig. 3b**). Three responses fit into this category.

The following is an example of the third type of response:

> The problem is you can provide a counterexample where you can add two positive numbers on one side but subtract one from the other on the other side. And the sum of two positive numbers will always be greater than subtracting one from the other.

This response expresses a reasonable idea, but the approach is a bit misleading, inaccurate, or incomplete because y would need to be negative for $x - y$ to be equivalent to the sum of two positive numbers. The student might have noted this condition but did not explicitly mention it. I placed seven students' responses into this stack of unclear, incomplete, or inaccurate observations. (In hindsight, I wish that I had asked each student in this category to clarify his or her response. I may have overlooked some important thinking.)

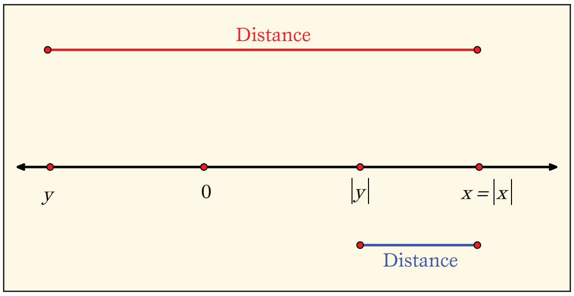


Fig. 4 The two distances can be different if $|x| > |y|$.

Although the approaches represented in **figure 3** are related, these two types of responses are different enough that they could lead to entirely different fixes. This possibility intrigued me, so during the next class period I asked students to develop a fix based on observations from the previous task.

To set the stage, I presented the two responses shown in **figure 3** to the rest of the class and facilitated a discussion about the two types of reasoning. Because Pat's response positions x and y in a manner such that $|x| - |y|$ is negative (see **fig. 3b**), I offered students my graph (see **fig. 4**) to illustrate that the distances Pat discusses could be different even when $|x| > |y|$.

I summarized the discussion by pointing out the following:

1. The right side of the inequality could be negative, which could happen even if x and y are both positive.
2. The distance between x and y could be greater than the distance between $|x|$ and $|y|$ even when both distances are positive.

I then asked students to work in pairs or groups of three to form propositions.

SOME PROPOSITIONS

After the discussion, students produced several propositions ("fixes"). In our classroom's vocabulary, these fixes were propositions, not conjectures, because students were asserting that their statements were true. That evening, I sorted the assertions into five categories:

1. If $|x| \geq |y|$, then $|x - y| \leq |x| - |y|$.
2. If $x \geq 0$ and $y \geq 0$, then $|x - y| \leq |x| - |y|$. (One group included the additional hypothesis that $y < x$.)
3. For all real numbers x and y, $|x - y| \geq |x| - |y|$.
4. If $x \geq 0$ and $y \geq 0$, then $|x - y| \geq |x| - |y|$.
5. For all real numbers x and y, $|x - y| \geq ||x| - |y||$.

During the next class period, I wrote all five propositions on the board, and we compared the fixes. We discussed how propositions 1 and 2 alter the hypothesis, how propositions 3 and 5 alter the conclusion,

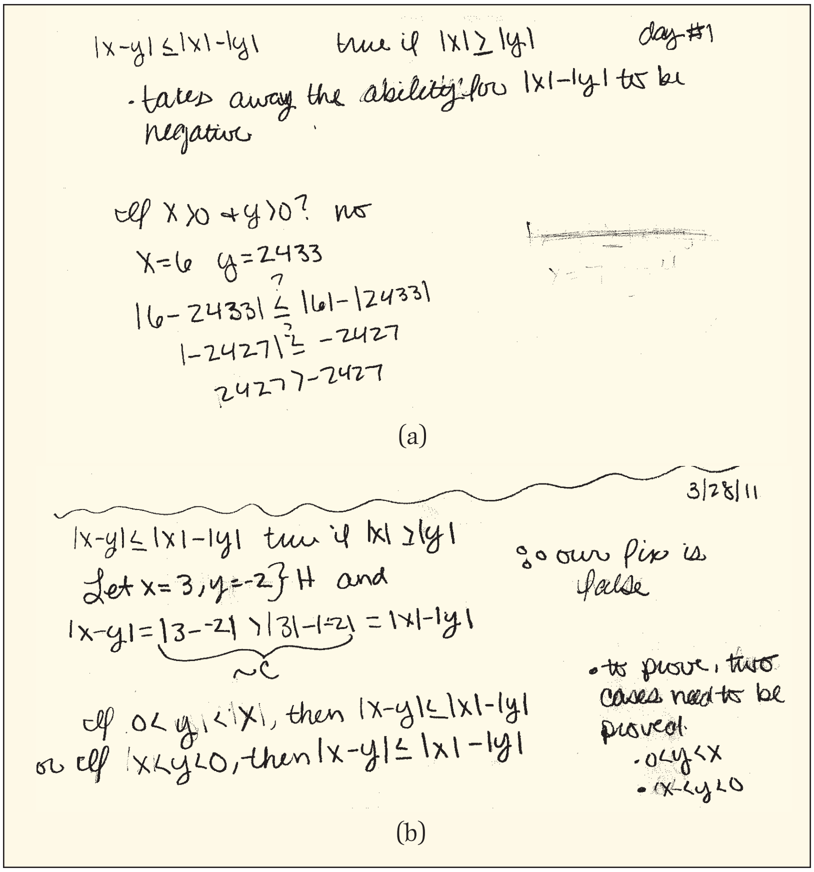


Fig. 5 Students discover an error (**a**) and try to correct it (**b**).

and how proposition 4 alters both. I then asked students to return to their groups (the same in which they had made the propositions) and repeat the original process. They were to either prove their proposition or provide a counterexample. Selected student responses are presented here. The first part of each figure referenced is the students' fix to the original conjecture; the second part is the students' evaluation (proof, counterexample, etc.) of their proposition.

Ashley and Dakota tell us that their fix attempts to resolve the question of the conjecture's sign (see **fig. 5a**). Then they show us that their fix is false and propose a different fix (see **fig. 5b**). These students are engaging in sense making because they use existing knowledge and prior observations in forming their proposition.

Alex and Casey also engage in sense making by grounding their fix in reasoning about distance (see **fig. 6a**). Then they prove a different result (see **fig. 6b**). We wonder whether they discovered that their fix in **figure 6a**, adding the hypothesis that $0 \leq y < x$, is trivial.

Jamie and Jean make an argument that could qualify as a proof with a few additional words or symbols (e.g., logical connectives and acknowledgment of prior results) (see **fig. 7a**, top left corner). Yet, in the right corner, they attempt another argument that is problematic. In the third line (labeled

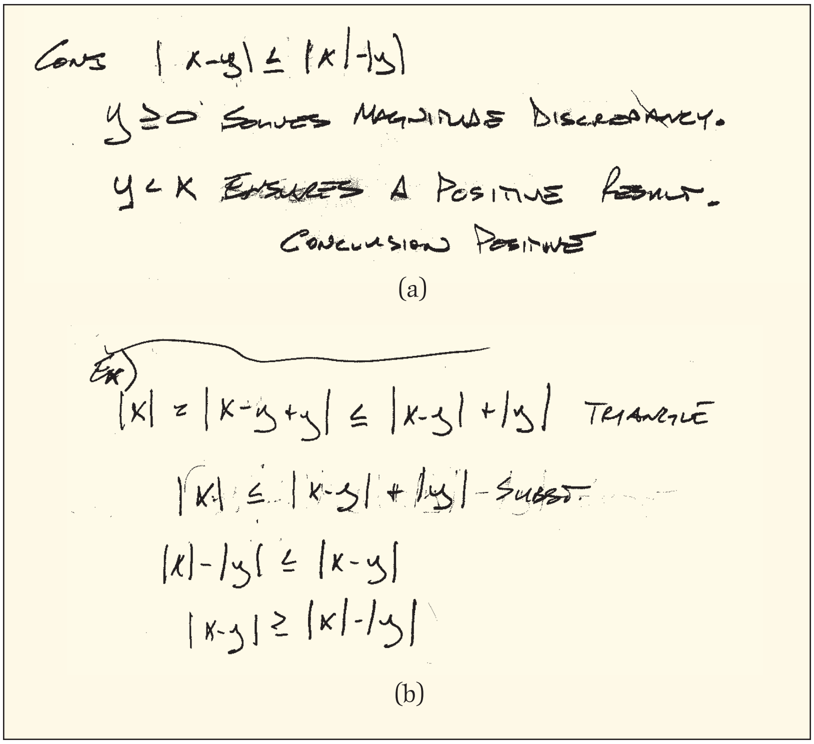


Fig. 6 Students use distance (**a**) to produce a different result (**b**).

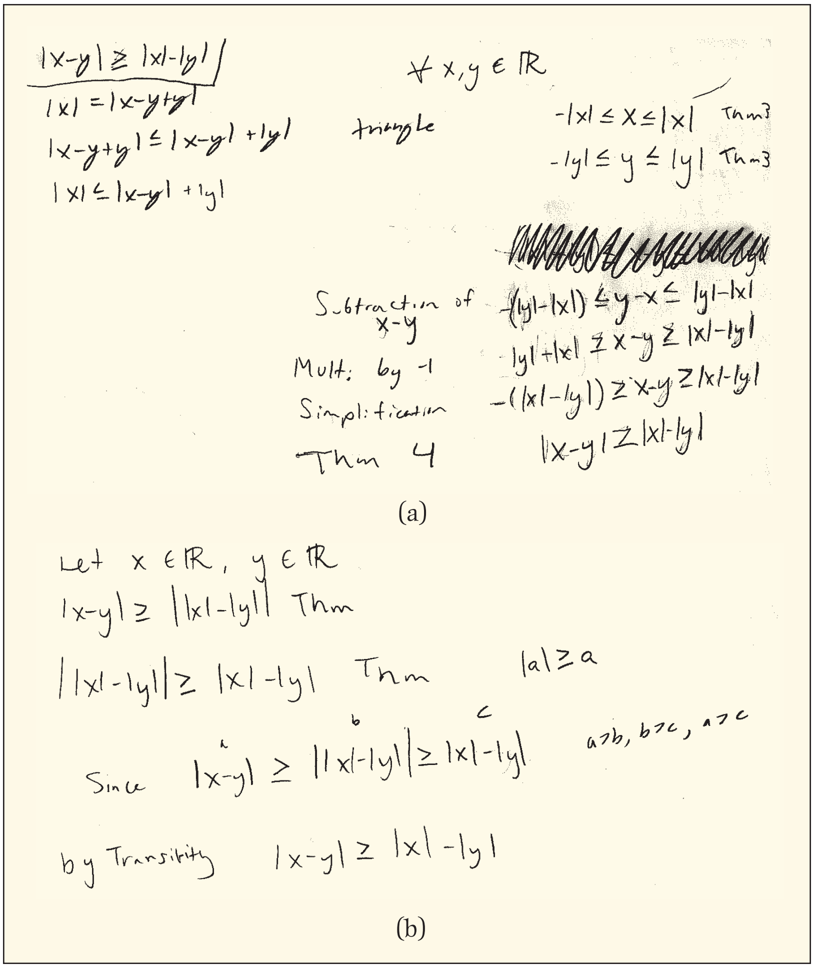


Fig. 7 Although some of the work in both (**a**) and (**b**) is correct, there is a reasoning flaw in (**a**).

"subtraction of $x - y$"), they misapply or inappropriately generalize the addition property of inequalities to subtraction. Also, the implication from the second-to-last line (labeled "simplification") to the last line is incorrect for two reasons: (1) we cannot assume that $-(|x| - |y|)$ is greater than zero; and (2) the property that the students appear to use, $0 \le |a| \le c \leftrightarrow -c \le a \le c$, is used incorrectly, even if they assume that $|x| - |y|$ is greater than zero.

Jamie and Jean write a third argument (see **fig. 7b**). This argument is correct, and it uses a very strong result in step 2. (This result was proven in class between these two activities.)

Jamie and Jean's arguments generated considerable classroom discussion. Although students valued both arguments, some preferred the last argument because it seems more sophisticated (e.g., it uses a stronger result). However, other students preferred the first argument because, in their words, "it better demonstrates why it is true," and it appeals to basic inequality principles.

The last example of student work presented here (see **fig. 8**) demonstrates how one group that conjectured about the fix in proposition 2 proves a different result (proposition 3) and then reconsiders the fix in an attempt to find a more appropriate hypothesis. Like Ashley and Dakota, Skylar and Jerry also model the intertwining of informal and formal reasoning as they make sense of the situation (NCTM 2009).

The fifth type of proposition listed above was a theorem stated in the text as a conjecture and proven in class while this activity was ongoing. (Jamie and Jean appeal to this result in the third argument for their proposition.) When the students who produced proposition 5 were asked to prove it, they mimicked the proof given in class. I did not criticize this approach; instead, I valued this work because mathematicians often appeal to results that they already know to be true, and they have a large repertoire of proof strategies and known proofs that they use.

REFLECTION

For this final section, I have chosen to write a reflection rather than a conclusion. To call this a conclusion would express intent other than the spirit in which this article is written. My goal as an educator is to develop and implement learning experiences for my students that reflect current understanding of mathematical teaching practice. During this activity, my goal was to facilitate students' learning about the interplay between informal and formal reasoning as well as to help them develop a "mathematical habit of mind" that involves a search for truth when faced with falsehoods. To say that I draw conclusions from this one experience would misrepresent my understanding of how mathematical best practices develop. Instead, I wrote this article to share my practice and to contribute to dialogue about high-quality, reasoning-based learning environments.

I now present some aspects of this activity on which I have reflected. First, I felt that students had an experience that promises to develop their understanding of mathematics as a discipline. We examined a conjecture, proved it false, examined the reasoning behind our counterexample, used this reasoning to develop some propositions, and then examined our propositions for truth values. This process exposed students to the nature of mathematical investigation, which includes examining, investigating, conjecturing, proposing, and reexamining in a continual quest for sense making. Structuring the activity as a sequence of three episodes over a two-week period exposed students to the concept that mathematics is not a lecture-homework-study-exam experience but, instead, a continuous and personalized quest for truth.

The fact that the activity began with a counterexample is also important. Too often, we demonstrate a counterexample and move on, satisfied that we have evaluated the claim. Logically, this is true; mathematically, however, the work of sense making has just begun. Students should learn that "doing mathematics" often involves learning from false starts.

Students also had opportunities to experience roles of proving and reasoning beyond establishing truth. Counterexamples do not have to explain why. However, students saw that when there is reasoning behind the counterexamples, opportunities for more learning emerge. In this sense, our proving was related to problem solving (e.g., solving the problem of writing a true statement that salvages as much of the initial conjecture as possible).

The reasoning and proving activities also helped students discover new mathematical propositions not in their textbook, and in some cases they even created new mathematics for themselves. They used their counterexamples (proofs of "there exist" statements) to develop and discover a new mathematical statement (e.g., for all x and y, [blank] is true).

Last, through reasoning and proving, the students had opportunities to learn about logic, absolute value properties, and absolute values and real number line distances, and they became more aware and appreciative of the work of mathematicians.

Students also observed the value of multiple solutions in light of their individual contribution to our mathematical understanding at every stage of the endeavor. Different counterexamples arose from different types of reasoning, which led to very different fixes, and fixing false conjectures involved altering hypotheses, conclusions, or both.

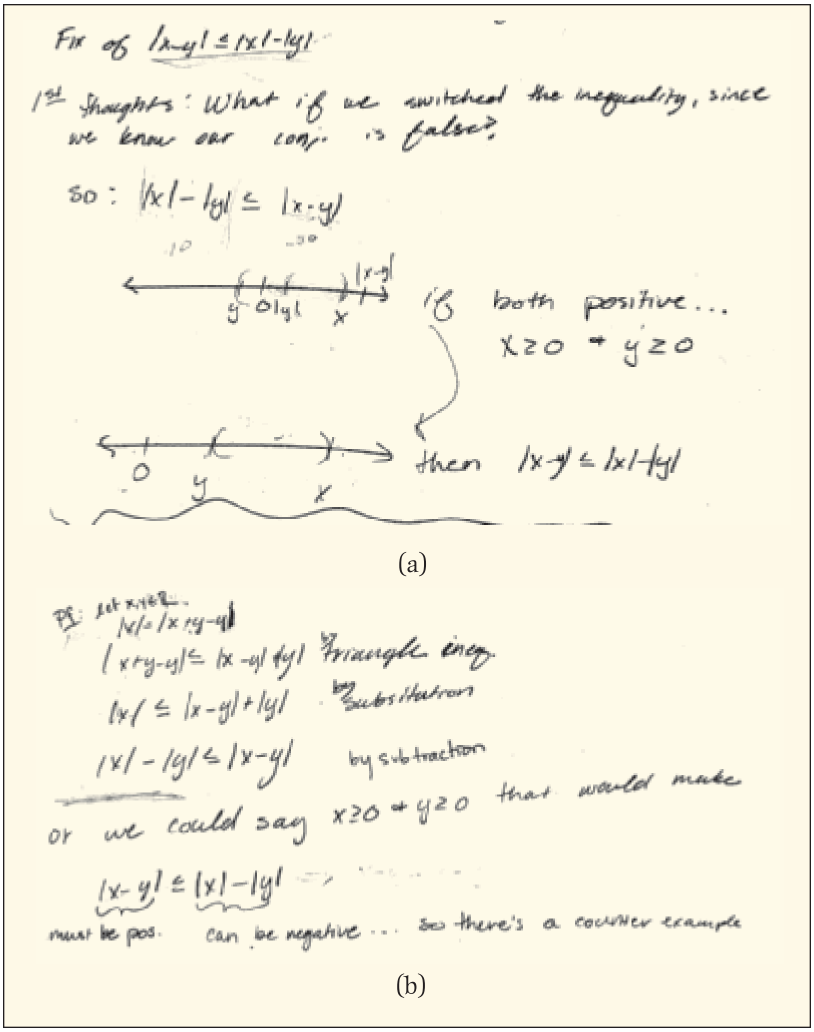


Fig. 8 Students whose work is shown in (**a**) and (**b**) use both formal and informal reasoning to prove proposition 3.

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