

Investigating Extrema with GeoGebra

Technology can be used to manipulate mathematical objects dynamically while also facilitating and testing mathematical conjectures. We view these types of authentic mathematical explorations as closely aligned to the work of mathematicians and a valuable component of our students' educational experience. This viewpoint is supported by NCTM and the Common Core State Standards for Mathematics (CCSSM).

NCTM's Technology Principle, for example, states: "Technology enriches the range and quality of investigations by providing a means of viewing mathematical ideas from multiple perspectives" (NCTM 2000, p. 25). Similarly, the first of CCSSM's eight mathematical practices—"Make sense of problems and persevere in solving them"—encourages teachers to work toward developing stu-

dents who "check their answers to problems using a different method" (CCSSI 2010, p. 6). In the fifth mathematical practice—"Use appropriate tools strategically"—proficient mathematics students are described as understanding that "technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data." (p. 7). In addition, CCSSM states that students should recognize the help that mathematical tools can offer as well as their limitations.

Here we discuss an exploration that we have used in the classroom with preservice secondary school teachers. We believe that it provides an excellent venue for students to experience the use of technology consistent with both NCTM's Technology Principle and CCSSM's mathematical practices. Specifically, it allows them to explore mathematics in multiple representations (algebraic and graphical) while experiencing both the benefits and the limitations of a powerful mathematical tool.

The exploration we present is based on an exploration suggested in NCTM's discussion of its Algebra Standard (NCTM 2000):

... students should learn to recognize how the values of parameters shape the graphs of functions in a class. With access to computer algebra systems (CAS) ... students can easily explore the effects of changes in parameter as a means of better understanding classes of functions. For example, explorations with functions of the form $y = ax^2 + bx + c$ lead to some interesting results. (p. 299)

The activity outlined in this excerpt—exploring how the graph of a quadratic equation (i.e., $y = ax^2 + bx + c$) changes as the parameters a , b , and c vary—is a rich and interesting investigation that has been discussed in earlier *Mathematics Teacher* articles (Edwards and Özgün-Koca 2009; Fallon and Luck 2010). We extend this exploration by investigating higher-degree polynomials. Although our work is with preservice secondary school teachers, we believe that this activity is appropriate for high school students in precalculus and calculus. Further, components may be suitable for students in an advanced algebra course. In the interest of thoroughness, we will work through the activity as a calculus task.

PROBLEM SETUP

To extend the original investigation to higher-degree polynomials, we must address an immediate question: Which point or points on the polynomial should we track as a parameter changes? The original task concerns tracking the location of the vertex of a quadratic. In the case of a quadratic, the location of the vertex is a unique point on the graph that represents a *local extremum* (i.e., a maximum or a minimum), the coordinates of which can be found by setting the first derivative equal to zero and solving.

If we extend this idea to higher-degree polynomials, finding the first derivative, setting it equal to zero, and solving will again help us find local maximums or minimums. However, unlike with the quadratic, these extrema will not necessarily be unique points (or, for that matter, may not exist at all). Thus,

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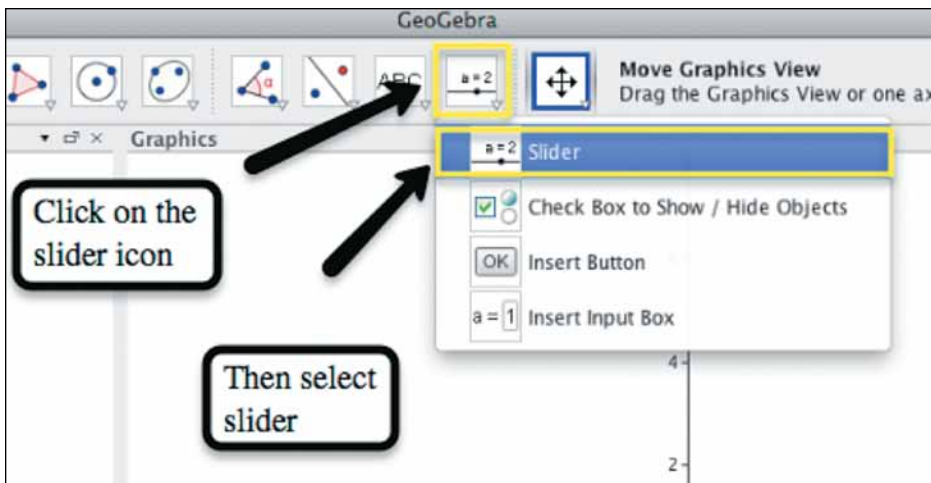


Fig. 1 Students click on the icon to create a slider.

simply answering the question can be a challenge. Because our aim is to engage students in mathematical exploration so that they might learn how to use technology to develop and test conjectures, we shall narrow our focus to the case of a cubic in standard form and consider the path traced by the extrema as we vary the parameters individually.

INVESTIGATING THE PATH OF EXTREMA

Using the dynamic geometry software GeoGebra (available for free from www.geogebra.org), we can easily construct a dynamic graph of any polynomial—for example, a cubic function in standard form, $y = ax^3 + bx^2 + cx + d$, where a is nonzero and where a, b, c , and d are real numbers. To do so, we will use the built-in slider tool. First, we click on the slider icon and then click on **Slider** from the drop-down menu (see fig. 1). We can now place a slider on the page by simply clicking on the page. Once we click on the page, a slider box will open up (see fig. 2), prompting us to input settings for the slider to be created. These settings include a name, the interval under consideration, and the increment. We have entered a for the name and left the default settings of **Min -5** and **Max 5** and increment **0.1**. Note that these settings can be adjusted later. After entering the settings, we click on **Apply** in the slider box, leaving us with a slider labeled a on our page. Repeating the process described, we create sliders for b, c , and d (see fig. 3).

Note that the sliders are located on the page where we clicked. To move

them, we first click on the selection icon in the upper-left corner. Next, hovering over one of the sliders and right-clicking will open up a menu showing the setting for the slider. Unchecking **Fix Object** will allow the slider to move when we click and hold the line segment (see fig. 4).

Now we enter the function that we want to explore. To do this, we type $a \cdot x^3 + b \cdot x^2 + c \cdot x + d$ into the input bar at the bottom of the screen and press **Enter** (see fig. 5). Note that if we type the expression as shown here, GeoGebra will automatically assign a function name. An appropriate function name, such as $h(x)$, can also be specified by students. The resulting graph will look similar to that shown in figure 6.

The name of this function will be defined in the algebra window on the left of the screen. In a new sketch, the function should automatically be named f . The parameters of the cubic can now be changed by manipulating the sliders with the mouse or by entering **parameter name=value** into the input bar. For example, to set the value of parameter b to 4, we would type $b=4$ into the input bar.

The extrema for the function can be constructed by using the built-in **Extremum** command. To construct the extrema of the cubic function f , enter **Extremum[f]** into the input bar. Note that in figure 1, $a = b = c = d = 1$, and there are no extrema. A document with $b = 4$ should have two extrema. Students and teachers can experiment and observe how changing the value of each parameter changes the location of the extrema.

We are now ready to consider the

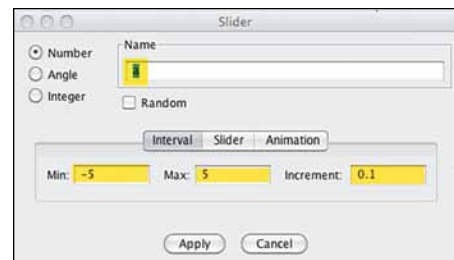


Fig. 2 Students set their slider inputs.

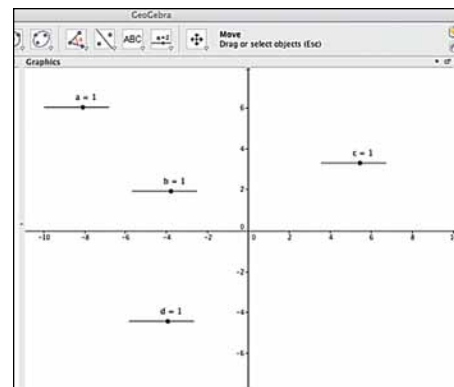


Fig. 3 The four sliders are displayed on the page.

primary focus of our exploration: What is the path of the extrema when each of the parameters a, b, c , and d is manipulated individually? Although students can develop conjectures using only observation, we have found it helpful to use the built-in trace feature to visualize the path. To trace a point, we right-click on the point and then left-click on the **Trace On** option. Once we have turned on the trace for the extrema, changing the value of a parameter results in a static representation of the path of the extrema displayed as a collection of plotted points. To erase the collection of traces, we right-click on either of the points and uncheck **Trace On**. Figure 7 shows the trace when a is varied.

After experimenting with all the parameters for a few minutes, we can generate several conjectures about the path of the extrema. A few examples follow:

- When parameter a is changed, the extrema appear to follow a quadratic path.
- When parameter b is changed, the extrema appear to follow a cubic path.
- When parameter c is changed, the extrema appear to follow a cubic path.
- When parameter d is changed, the extrema appear to follow a vertical path.

Although GeoGebra is useful for generating and testing conjectures, it is not helpful for generating the equations that model these paths. We will show how these equations can be generated using algebra and basic calculus concepts.

PROVING CONJECTURES

Let's first consider finding the path created by the extrema when parameter a is changed. As previously mentioned, the value of the derivative of f at each of the extrema points is zero. There-

fore, we can use the derivative to find an equation that relates a and x . Using this information, we can then find an equation that models the path of the extrema as a is changed. Beginning with the general form of the cubic, $f(x) = ax^3 + bx^2 + cx + d$, we set the first derivative $f'(x) = 3ax^2 + 2bx + c$ equal to zero and solve for a :

$$a = \frac{-2bx - c}{3x^2}$$

and

$$f(a) = f\left(\frac{-2bx - c}{3x^2}\right) = \frac{1}{3}(bx^2 + 2cx) + d.$$

The function f models the path followed by the extrema. Students can verify this by plotting f in GeoGebra and comparing it with the path created by the trace of the extrema. We enter the function above into the input bar in GeoGebra and then vary a to verify that the extrema do in fact fall along the path defined by $y = 1/3(bx^2 + 2cx) + d$ (see **fig. 8**). Using the same approach, we can find and verify the equations for the paths followed by the extrema when b and c are changed.

The equation for the path followed by the extrema when d is changed cannot be found using this method because $f'(x)$ does not depend on d . Instead, we need to shift our attention to the ordered pairs (x, y) that define the extrema. To find the ordered pair mentioned above, we set $f'(x) = 0$, solve for x , and then substitute this x -value into $f(x)$ to find the corresponding y -value. The resulting ordered pair has x -coordinate

$$\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right)$$

and y -coordinate

$$\left(a\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right) + b\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right) + c\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right) + d\right).$$

By focusing our attention on the fact that the expression for x does not

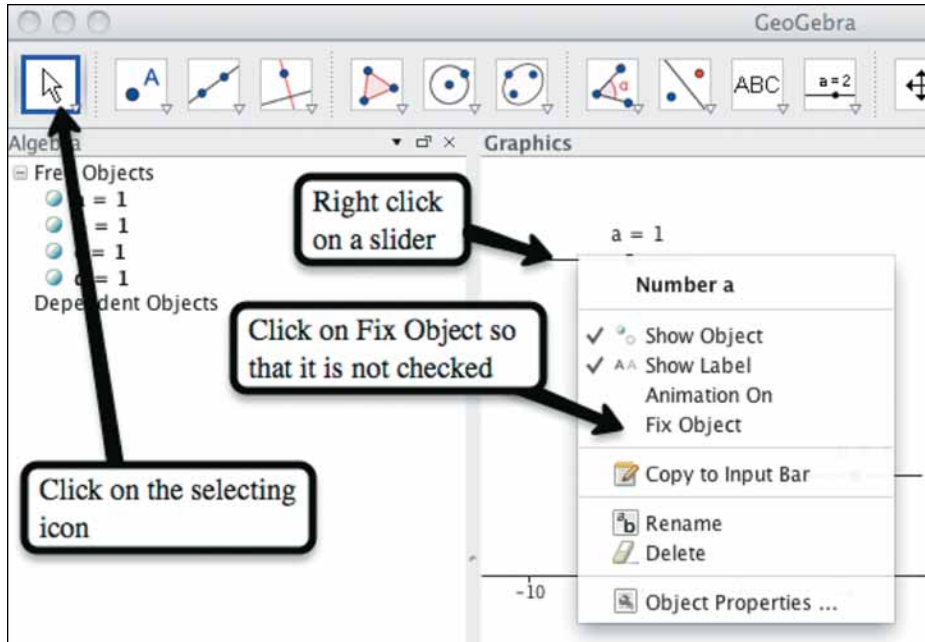


Fig. 4 Now the sliders are more properly aligned on the page.

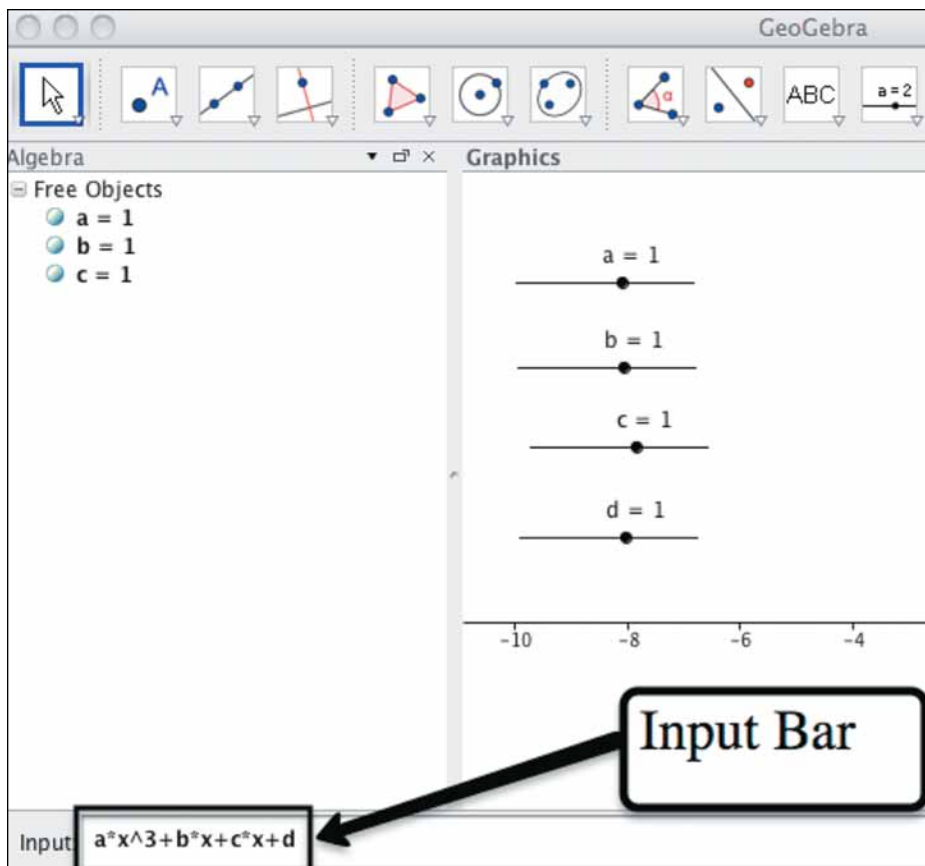


Fig. 5 The input bar is used to enter the cubic function into GeoGebra.

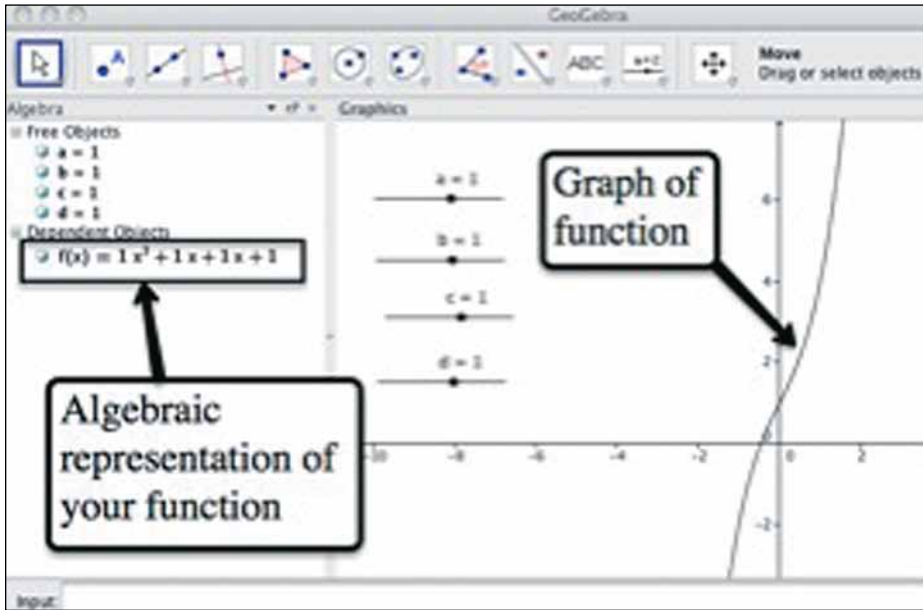


Fig. 6 The graph is now ready to be manipulated.

depend on d whereas the expression for y does, we can deduce that a change in the value of d will result in the extrema tracing the vertical line from the x -coordinate above. In other words, changing the parameter d does not change the x -coordinate of the extrema but does change the y -coordinate.

A more rigorous mathematical method to determine the path traced by the extrema when d varies is to analyze the effects of a change in parameter d on the x - and y -values of the extrema individually. Essentially, we notice that the derivative of x with respect to d is 0, whereas the derivative of y with respect to d is 1:

Thus, it can be shown that

$$\left(\frac{dy}{dd}\right) = \frac{dy}{dd} \cdot \frac{dd}{dx} = \frac{dy}{dx} = \frac{1}{0}$$

This illustrates that as d is changed, the extrema will trace a path that has an undefined slope—that is, a vertical line.

A REALISTIC MATHEMATICAL EXPERIENCE

Throughout this investigation, we see opportunities for students to engage in three important mathematical activities: generating, testing, and proving mathematical conjectures. The use of technology—in this case, GeoGebra—allows students to observe a phenomenon as it occurs. After clearly stating this observa-

tion and testing it under multiple conditions, students are typically convinced that their conjecture is true. With this “certainty” in mind, they are motivated to find the mathematical model, necessitating a shift away from the technology that they used to generate the conjecture.

Through this process, mathematically proficient students can engage in at least two of CCSSM’s eight mathematical practices. First (and most directly), students used tools strategically (practice 5) as they tried to make sense of problems and persevere in solving them (practice 1). Second, students had the opportunity to experience the benefits of the mathematical tool they used—in this case, GeoGebra—to explore and visualize while realizing the limitation of the tool

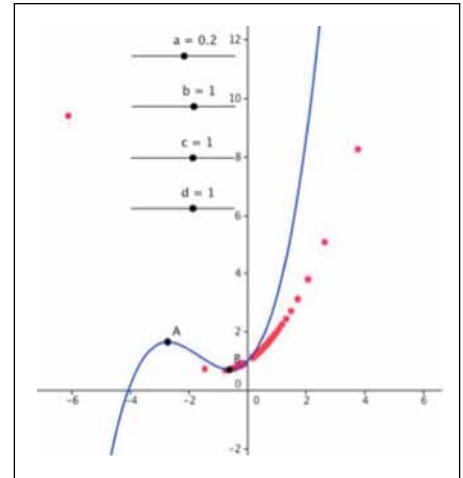


Fig. 7 This screen shot shows the trace extrema of the curve $y = ax^3 + x^2 + x + 1$ at points A and B as parameter a is varied.

when it came time to generate a specific rule for the path traced by the extrema. This process allows students to engage in a more realistic mathematical experience, including the discovery and generation of a conjecture and the resulting mathematical analysis.

The exploration described here, used in a technology-rich environment, provides an excellent opportunity for students to engage in a realistic mathematical experience. Throughout the experience, students work through all aspects of the proof process: exploration, conjecture, and proof (Burke et al. 2008). Although this particular exploration focused on the paths of the extrema of a cubic, it can be extended to higher-degree polynomials and other interpretations (e.g., tracking the extrema for the general quartic).

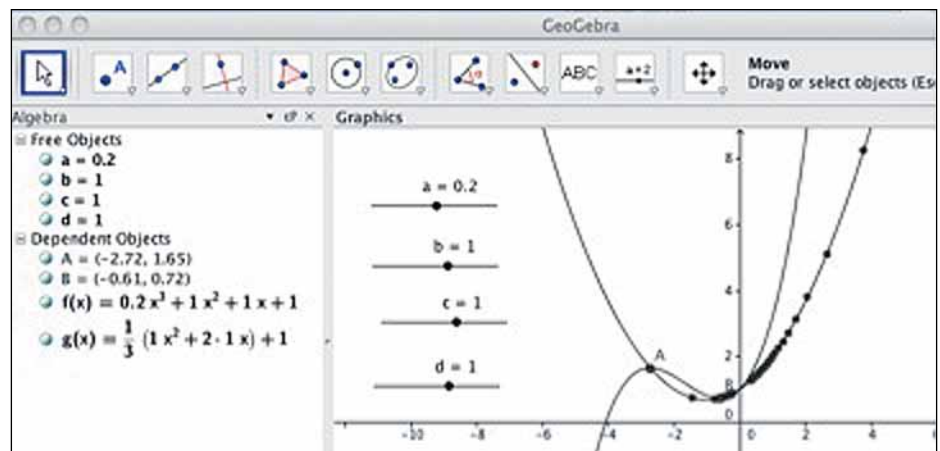


Fig. 8 This screen shot displays the graph of the original function, the collection of traced points, and the predicted path of the traced points.

On the basis of our initial work, we have generated several conjectures about the behavior of the extrema for any polynomial of degree n :

- For any polynomial of degree n with leading coefficient a_n , the path of the extrema when a_n is varied will be a polynomial of degree $n - 1$.
- For any polynomial of degree n with constant term a_0 , the path of the extrema when a_0 is varied will be a vertical line.
- For any polynomial of degree n with coefficients a_n, a_{n-1}, \dots, a_0 , the path of the extrema when any one of a_{n-1}, \dots, a_1 is varied will be a polynomial of degree n .

Proving (or disproving) these conjectures could be meaningful extensions of this investigation. We encourage educators and students to explore this activity and consider the potential it holds for bringing meaningful mathematics into the secondary school classroom.

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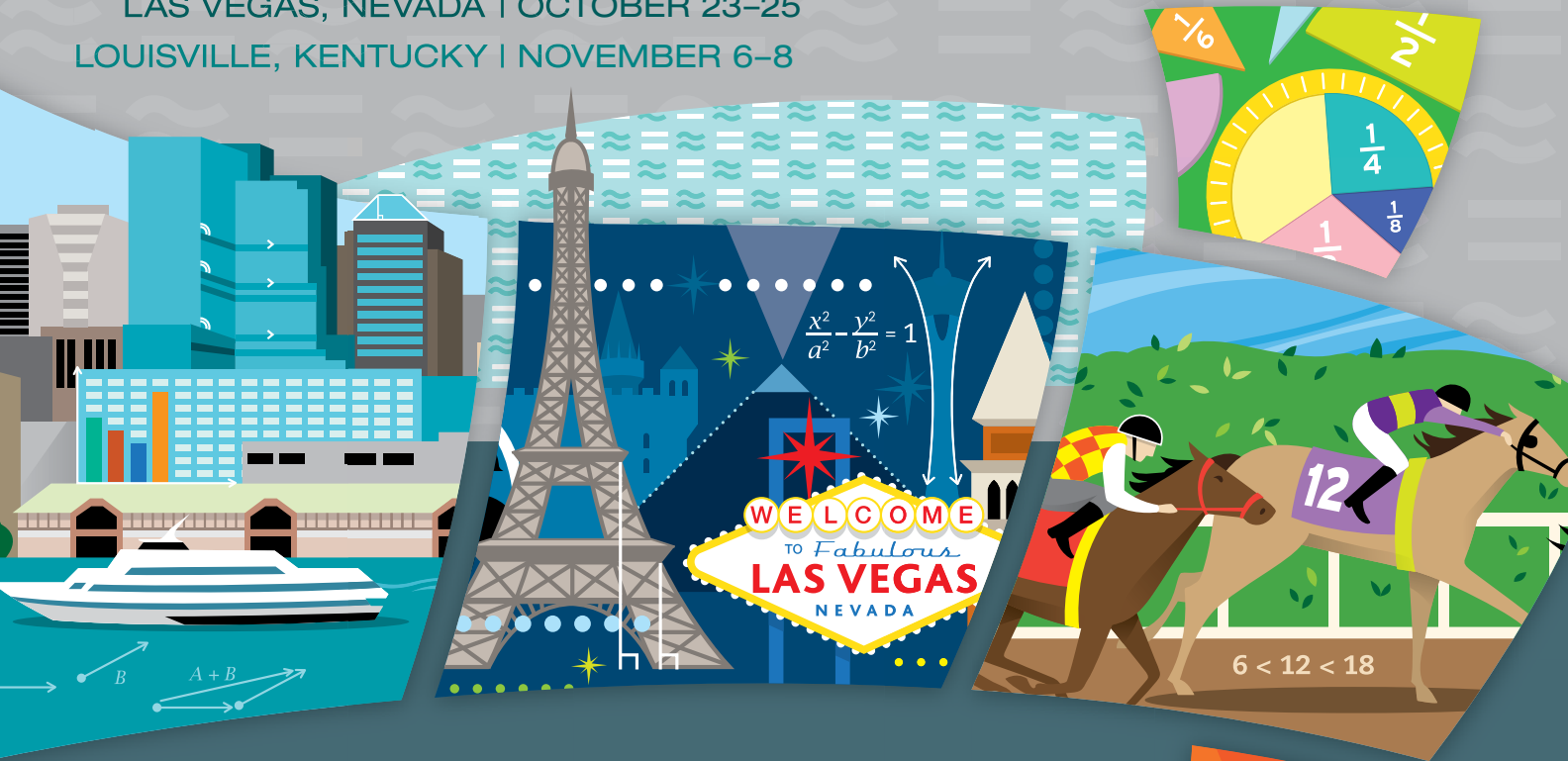
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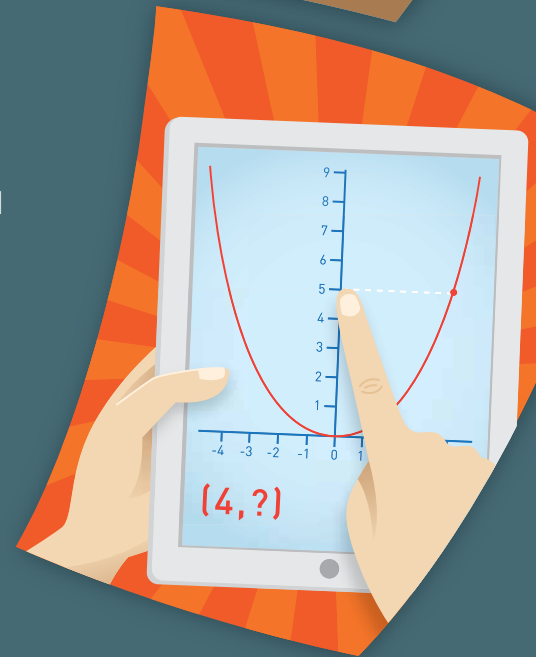


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