Shapedoku puzzles combine logic and spatial reasoning with an understanding of basic geometric concepts.

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Shapedoku is a new type of puzzle that combines logic and spatial reasoning with understanding of basic geometric concepts such as slope, parallelism, perpendicularity, and properties of shapes. Shapedoku can be solved by individuals and, as demonstrated here, can form the basis of a review for geometry students as they create their own. In this article, we trace the evolution of Shapedoku puzzles from their Sudoku ancestry, describe how one high school teacher introduces them into her geometry classroom, and highlight some high school students' work as they reason through their own creations. Finally, we provide activity sheets of puzzles created by high school students for teachers to use in their own classrooms.

BACKGROUND

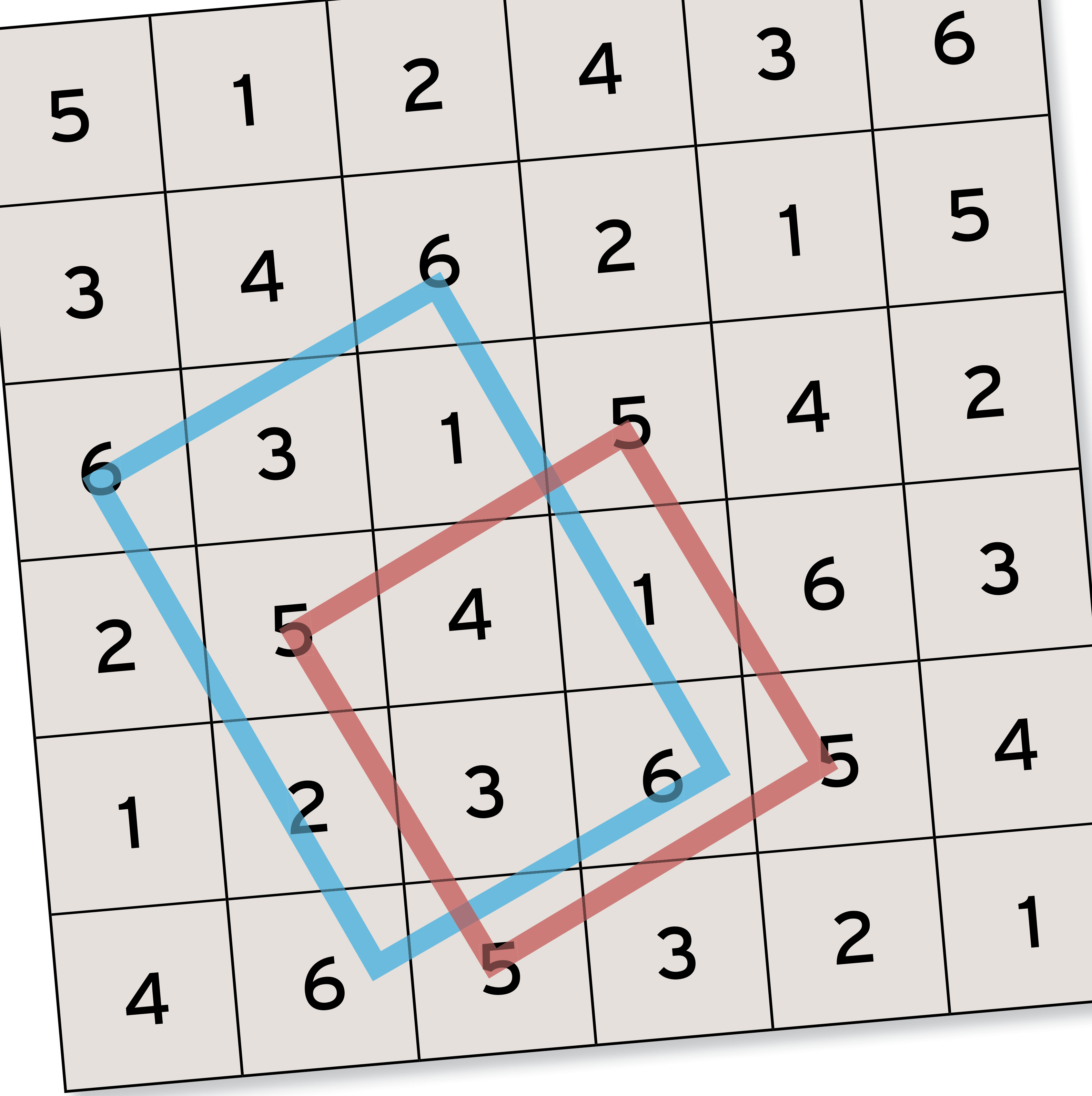
A standard Sudoku puzzle begins with some numbers placed in a square grid and asks the solver to complete the grid by placing numbers so that no number repeats in a row, column, or outlined region (see **fig. 1**).

Hundreds of variations of Sudoku puzzles exist (see Pegg [2005] for a few examples). All Sudoku variations use deductive and spatial reasoning in finding solutions, but some draw on and support other mathematics. For example, Killer Sudoku puzzles require solvers to use basic addition and subtraction facts to find solutions. In these puzzles, solvers are given the sums of the distinctly different numbers that lie in a region bounded by a dotted border. No starting numbers are typically placed in the Killer Sudoku grids (see **fig. 2a**).

While trying to create a professional development session, author Wanko and one of his undergraduate students were searching, without much luck, for a Sudoku variation that uses some aspect of geometric thinking. However, they noticed that the placement of numbers in a Sudoku solution indicates some basic geometric shapes. For example, four of the 5s in the 6×6 Sudoku solution (see **fig. 1b**) outline the vertices of a square, and four of the 6s outline the vertices of a parallelogram (see **fig. 3**). They realized that a Sudoku variation could be created that draws on the solver's knowledge of attributes of geometric shapes. Thus, Shapedoku was born!

After working out some of the kinks involving the number of clues that would be needed and what shape names would be used, Wanko and his colleague introduced Shapedoku puzzles to teachers in 2009. Since then, a number of teachers have incorporated these puzzles into their mathematics classrooms—to teach and reinforce basic shape properties and to challenge students to apply their reasoning and sense making in a new and different context.

Reinforcing Geometric Properties with Shapedoku



Puzzles

SHAPEDOKU RULES

Shapedoku puzzles have been created on 4 × 4, 5 × 5, and 6 × 6 grids. Unlike Sudoku puzzles, Shapedoku puzzles do not have additional outlined regions (such as the 3 × 2 boxes in the 6 × 6 Sudoku puzzle in **fig. 1**). Thus, Shapedoku puzzles are technically not a variation of Sudoku but a variation of Latin squares, puzzles with the only condition that each number appears once in each row and each column.

Each starting Shapedoku grid includes some numbers that are already placed (see **fig. 4**). These numbers are circled so that they can be easily distinguished from other numbers as they are placed in the grid. In addition to these starting numbers, solvers are provided with a list of shapes that are created by connecting the centers of the squares containing the noncircled numbers in the solution. Note that the starting circled numbers are not used in forming the indicated shapes. Thus, in a 5 × 5 Shapedoku puzzle, for example, a type of quadrilateral is created when there is one specific circled number, and a type of triangle is created when there are two specific circled numbers. For example, in the puzzle shown in **figure 4**, the two circled 4s imply that the missing 4s lie in the shape of a triangle.

The final rule in solving Shapedoku puzzles is an extremely important one: The name that is listed is the most specific name that can be used to describe the shape. For example, a square shape would be listed as a *square* and not as a rhombus, parallelogram, rectangle, or quadrilateral. And a shape that is listed as an *isosceles triangle* must have exactly two sides of the same length and cannot contain a right angle (it would then be called an *isosceles right triangle*).

Because definitions of geometric shapes vary widely across textbooks used in the United States, we decided to focus on definitions that are used in some of the more common textbooks, including the one that Nickell used in her classes. Shapedoku puzzles can be adapted to align with any set of definitions that students are learning and using, but the ones shown here focus on classifying *triangles* on the basis of side lengths and the existence of right angles (topics covered in Nickell's curriculum) and defining *trapezoids* as quadrilaterals with exactly one pair of parallel sides (matching the same textbook's definition). We recognize that this "exclusive definition" is not universally accepted in the United States (Usiskin and Griffin 2007), but it supports Nickell's students' conception of a trapezoid. A classification diagram of the thirteen different shapes used in these Shapedoku puzzles is given in **figure 5**.

In this version of Shapedoku, *trapezoids* can be further classified as right or isosceles trapezoids. *Quadrilaterals* are polygons with four sides and no other distinguishing characteristics, although a quadrilateral could have exactly one right angle

	1				**6**
			2		**5**
6			**5**		
		4			**3**
1		**3**			
4				**2**	

(a)

5	1	2	4	3	6
3	4	6	2	1	5
6	3	1	5	4	2
2	5	4	1	6	3
1	2	3	6	5	4
4	6	5	3	2	1

(b)

Fig. 1 A 6 × 6 standard Sudoku puzzle grid (**a**) contains some numbers used as hints for discovering the solution (**b**).

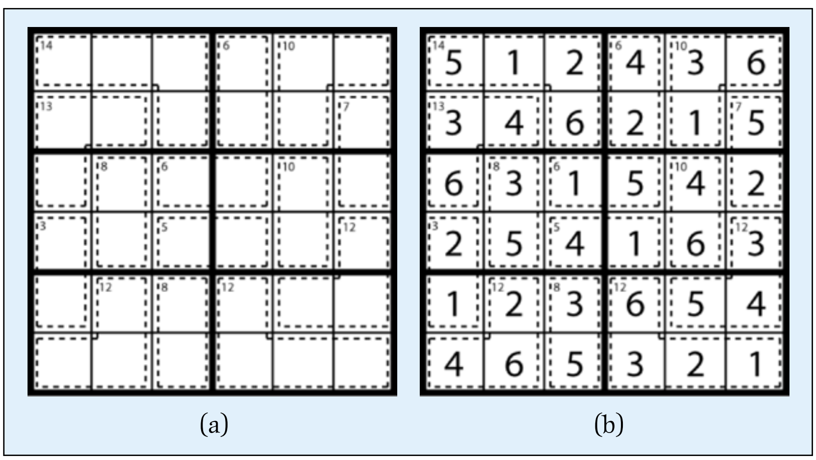


Fig. 2 Killer Sudoku puzzles use additional regions outlined with dotted borders and indicate the sum of the numbers in the regions.

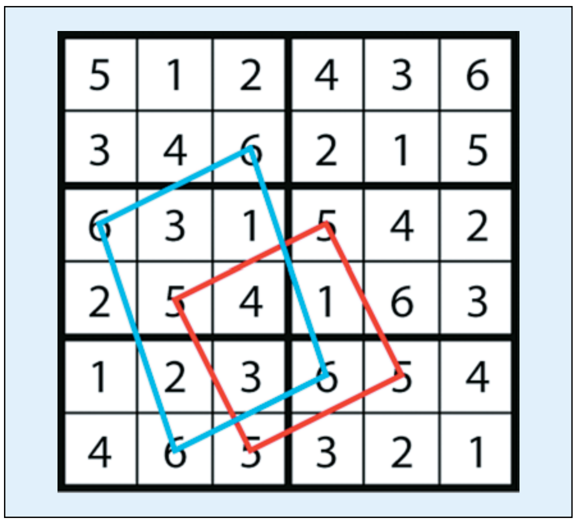


Fig. 3 In the Sudoku solution, some 5s can be connected to create a square, and some 6s can be connected to create a parallelogram.

(*right quadrilateral* is not a commonly accepted shape name). The shape definitions that are used here are not the only ones that could be used in creating Shapedoku puzzles. Readers are encouraged to adapt Shapedoku puzzles to match their own curricula and classroom needs as necessary.

Solution grids do not typically include overlays of the shapes, because they would overlap and would be hard to distinguish. For clarity, the shapes have been shown in the solution to the Shapedoku example (see **fig. 6**).

Because each number appears exactly once in each row and each column, none of the sides of the indicated shapes is oriented horizontally or vertically. For this reason, shapes can be a little more difficult to visualize when solving Shapedoku puzzles. Students might consider slopes of line segments when identifying parallel and perpendicular sides and consider whether sides are congruent. For example, see the parallelogram using the 2s and the isosceles right triangle using the 4s in **figure 6a**.

IN THE CLASSROOM

Author Nickell, who taught ninth-grade honors geometry at Lakota West Freshman School in West Chester, Ohio, is one teacher who has incorporated Shapedoku in her classroom. Nickell attended Wanko's 2011 NCTM presentation, which highlighted several different types of logic puzzles that support various geometry topics.

During the session, Nickell began working through the Shapedoku puzzles with group members. They had conversations about how to start solving the puzzle. One teacher always started by using "Sudoku logic"—that is, making sure that each number appears once in each row and column. Another teacher started by trying to place a polygon that could have only one location in the grid given the properties of the polygon. The group members discussed parallel and perpendicular segments, congruent side lengths, and use of the most descriptive name of a polygon.

After listening to the conversations taking place and working through the activity, Nickell knew

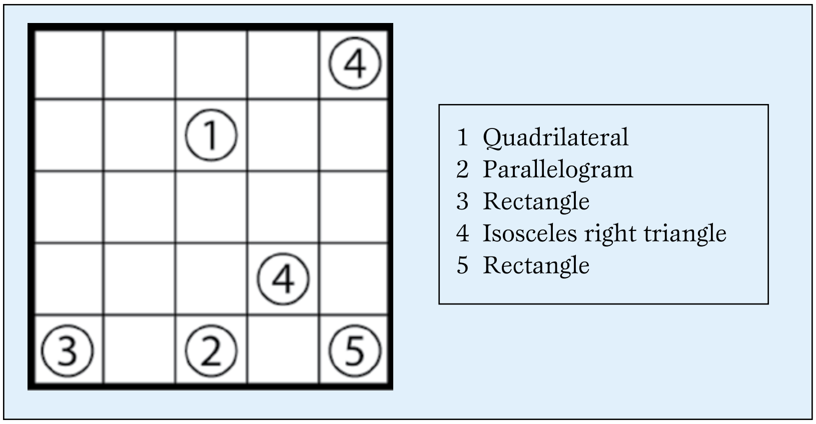


Fig. 4 The 5 × 5 Shapedoku example includes the names of the shapes that are created by connecting the noncircled numbers in the solution grid.

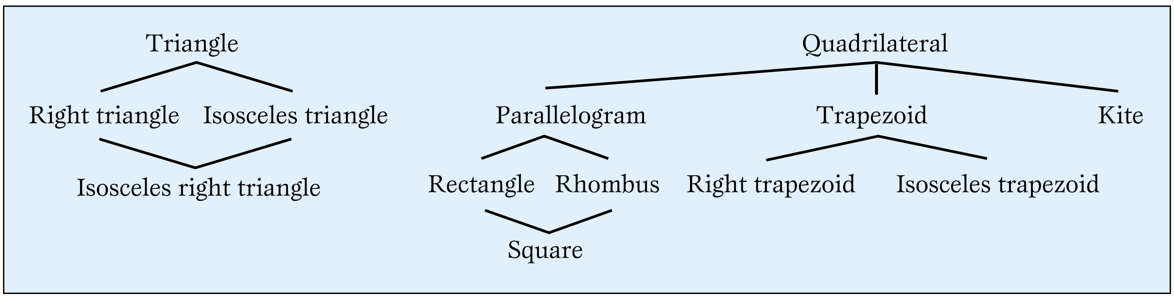


Fig. 5 The thirteen shapes used in Shapedoku are classified with the most descriptive names given toward the bottom.

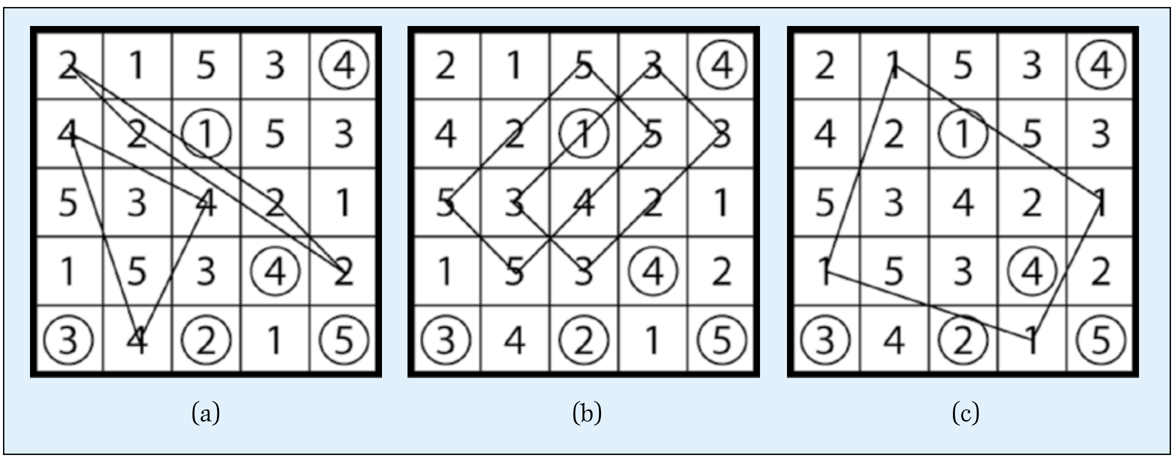


Fig. 6 The five indicated shapes are outlined in separate solution grids.

immediately that this would be an excellent activity to use with her students. Her hope was that her students would use their higher-level thinking skills to work through the puzzles and participate in discourse similar to what she had experienced in this group.

Nickell used Shapedoku in her classroom as a supplemental activity to reinforce the properties of polygons, specifically special quadrilaterals. She incorporated it for three days after the coordinate geometry chapter in her honors geometry classes. The students had spent three weeks studying the distance formula, midpoint, slope, vectors, and coordinate geometry proofs.

On the first day, Nickell introduced Shapedoku puzzles to her classes. She provided students with the Shapedoku rules and showed them a blank Shapedoku grid and the corresponding answer key. Then students discussed the basic Sudoku rules and how they applied to this puzzle, using the most descriptive name of each shape, and how to identify the shapes using parallel and perpendicular slopes. Once the students understood the basics of the puzzle, Nickell distributed some 4×4 and 5×5 puzzles for students to work on in small groups (taken from Wanko [2010]).

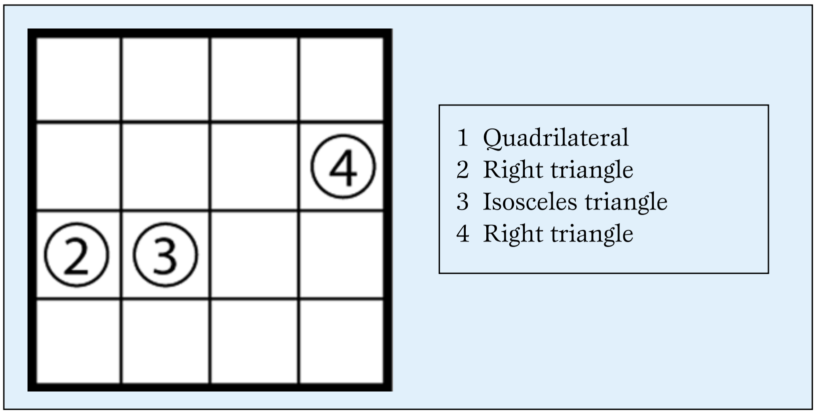


Fig. 7 Josh created a 4×4 Shapedoku puzzle that Alaina worked to solve.

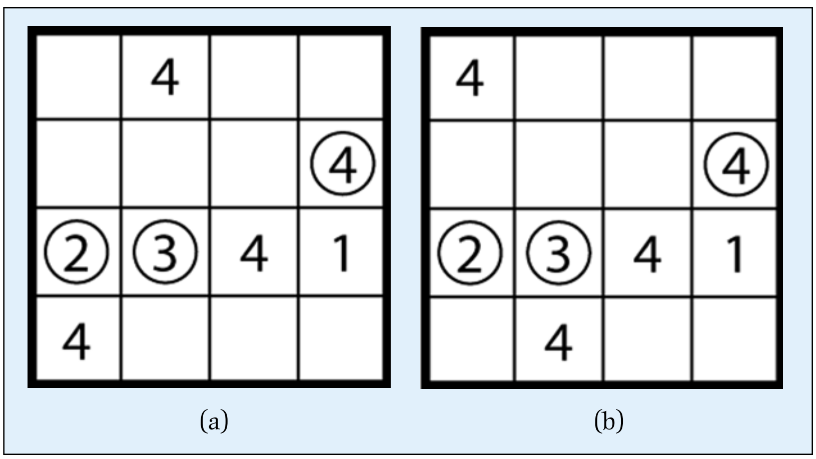


Fig. 8 Alaina made two possible placements of 4s on Josh's puzzle.

Students began by filling in numbers using Sudoku logic—trying to solve the puzzles entirely without using the list of shapes—but then would get stuck. They then resorted to guessing where to fill in numbers or using, in their words, a "gut feeling." Nickell asked students which shapes were usually helpful and not helpful to start with. Students came to a consensus that the rectangle and the square were beneficial starting points because of, in the case of the square, the requirements of perpendicularity and congruent sides.

While students were working, Nickell circulated around the room, pushing students to participate in the same discourse that she participated in at the NCTM conference. She asked each group to communicate why and how it chose the locations of the numbers. In doing so, students were required to discuss the slopes of the shapes to justify their decisions.

On the second and third days, Nickell challenged students to go beyond just solving Shapedoku puzzles to creating their own. The assignment was for each student to create a 4×4 and a 5×5 puzzle. The 5×5 puzzle also had the requirement that it must contain at least three different quadrilaterals. Without that requirement, many students wanted to make a 5×5 puzzle with all triangles, a condition that did not align with the Standards or Nickell's goal for the lesson. The in-class activity was organized into three parts (similar to writing a paper): composing a rough draft, having it edited by peers, and creating a final draft. She provided students with blank 4×4 and 5×5 grids for creating their rough drafts.

To create the Latin square, students used two different methods. Some students inserted numbers by trial and error until eventually a Latin square was formed. Others wanted more control over what shapes would be in their puzzle. Instead of randomly inserting numbers, they tried to place all occurrences of one number to form a specific shape before moving on to the next number and the next, until the Latin square dictated where the remaining numbers must go. For example, one group wanted its puzzle to contain a square and a rhombus. So these students began by first inserting the 2s into their Latin square to form a square and then circling the one 2 that did not create a square. Then they inserted the 3s to create a rhombus, circling the one 3 that did not create the rhombus. They filled in the rest of the Latin square using trial and error.

Once students thought that they had created a puzzle, they gave it to a partner to try to solve. The partner also checked to make sure that the solution was unique and double-checked that the most descriptive names were being used for the shapes. When that process was complete, students provided Nickell with the puzzles and their answer keys.

EXAMPLES OF STUDENTS' REASONING

As students created puzzles and their partners tested them for mathematical correctness and uniqueness, they used a great deal of logical reasoning as they communicated about their puzzles.

Verifying Solutions

Josh had created a 4×4 puzzle (see **fig. 7**), and Alaina was trying to verify the solution. She quickly filled in the third row, using her Sudoku logic, but could not decide which shape she should try to place first. Josh prompted Alaina to think about which shape had the fewest numbers remaining to fill in and, thus, had the fewest possibilities. Alaina noted that the isosceles right triangle, created by the 4s, needed only two additional numbers, whereas the rest required three, so she decided to start there. She made two possible solution grids, looking at which columns and rows still needed 4s (see **fig. 8**).

Alaina commented, "Well, the 4s are supposed to make a right triangle. In each of these two possible solutions, the 4s make a right triangle, but since you didn't say that it is a right isosceles triangle, then it must be the second one on the right." As she began to cross out the grid on the left, Josh looked at his intended solution and stopped her: "Whoa, wait a minute." He looked at his solution again and said, "Um, I think I made a mistake." He inserted the word *isosceles* on his original problem (see **fig. 9**), commenting, "I meant to say 'isosceles right triangle' on the 4s." Alaina went on to solve the problem using her first solution option and verifying that Josh's solution was both correct and unique.

Not using the most descriptive geometric term possible to name a shape was a common problem for students. This oversight sometimes led to a puzzle having more than one solution or—as in Josh's case—no solution. (If Josh's original problem is followed to its completion, three 1s end up being collinear, making it impossible to create a quadrilateral.) Nickell noted that this was helpful to her in preparing her students for their final exams; she emphasized the need to attend to discerning between different shapes on the basis of their specific attributes.

Confirming Uniqueness

Another common student mistake was creating a puzzle that was not unique. Nate had created a 4×4 puzzle (see **fig. 10**), and Malik was trying to verify his solution. Malik knew from his previous work that there was only one possible location for the 2s, and he quickly filled these in to form a square (see **fig. 11**). Then he used Sudoku logic to place the 1 in the fourth column, allowing him to fill in the location of the rhombus. Malik used more Sudoku logic to fill in two remaining numbers in the bottom row but then encountered a problem.

Trying to ensure that Nate's puzzle had exactly one solution, Malik used the method that other students in the class typically used—exhaustion. That is, he looked at every possible arrangement that could exist at this point and checked to see whether any of these arrangements could be eliminated according to the shape names given in the original puzzle. Then he noted, "There are two possible ways to make a right isosceles triangle with

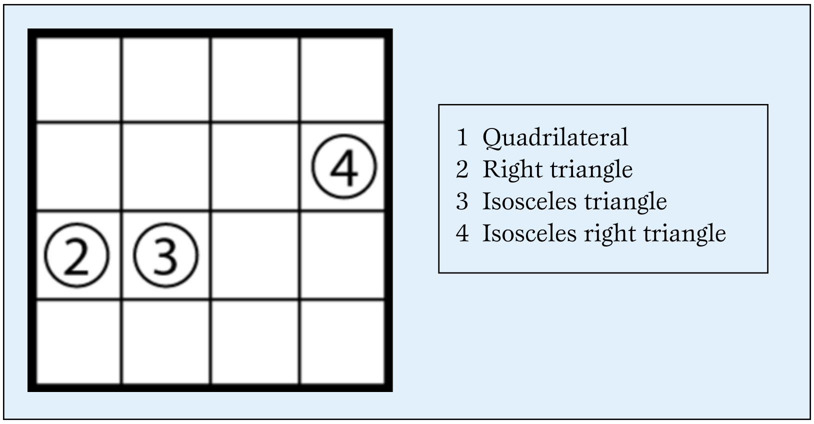


Fig. 9 Josh revised his 4×4 Shapedoku puzzle using Alaina's suggestions.

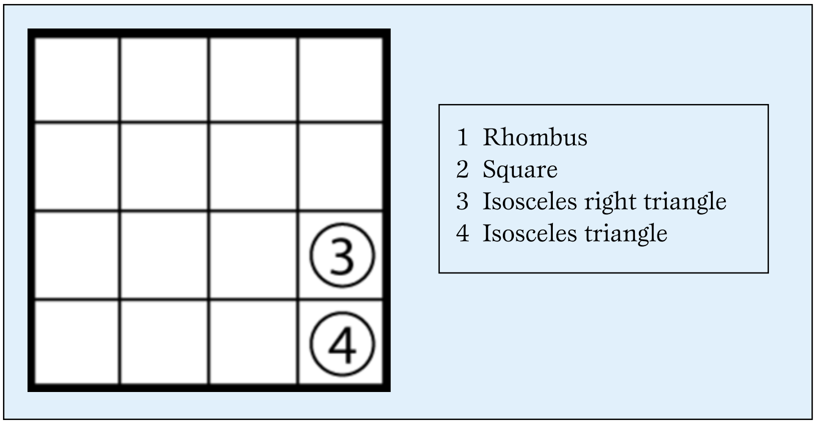


Fig. 10 Nate created a 4×4 Shapedoku puzzle that Malik worked to solve.

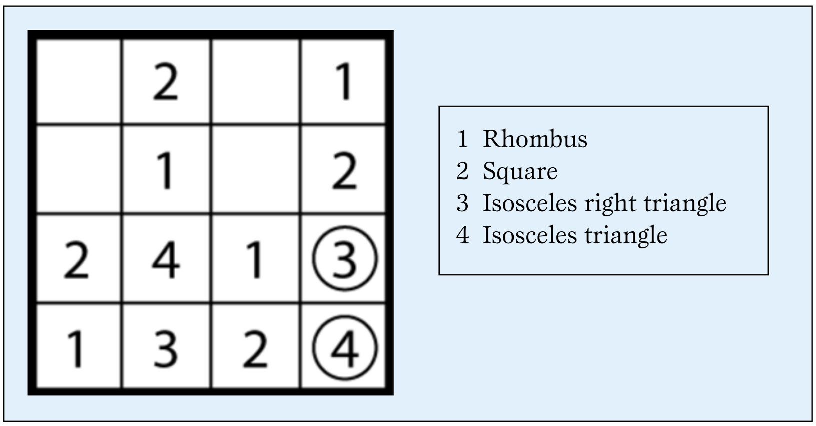


Fig. 11 Malik reached an impasse when solving Nate's puzzle.

the remaining 3s," and showed his paper to Nate, noting that both solutions also lead to an isosceles triangle with the 4s (see **fig. 12**). Nate examined his intended solution on his paper and immediately told Malik that he was wrong. They closely examined both solutions to verify that the most descriptive name was being used and came to a consensus that in fact Nate's puzzle had two solutions.

"This means I have to start all over!" Nate exclaimed. Nickell queried, "Could you find a way to change your current puzzle to make it unique?" Nate knew that the 1s and the 2s were fine; the 3s and 4s were causing the problem. He decided to uncircle the initial 4 and change its listing from an isosceles triangle to a quadrilateral (see **fig. 13**). Because all the 4s created a rhombus in one of the original possible solutions, this adjustment eliminated the problem and left Nate's puzzle with a unique solution.

HONING UNDERSTANDING AND PROBLEM-SOLVING SKILLS

As the students worked with their partners to create, test, and revise their puzzles, it became clear that writing their own puzzles was a critical part of their reasoning and sense making. In the Common Core State Standards, the third Standard for Mathematical Practice states that students should "construct viable arguments and critique the reasoning of others" (CCSSI 2010, p. 6). This activity provided students with a context for doing just that: They used their deductive reasoning skills to determine whether a solution is unique, and they worked with their partners to revise their puzzles to make sure that they were correct. By both solving and creating Shapedoku puzzles, students honed their understanding of geometric properties and their problem-solving skills.

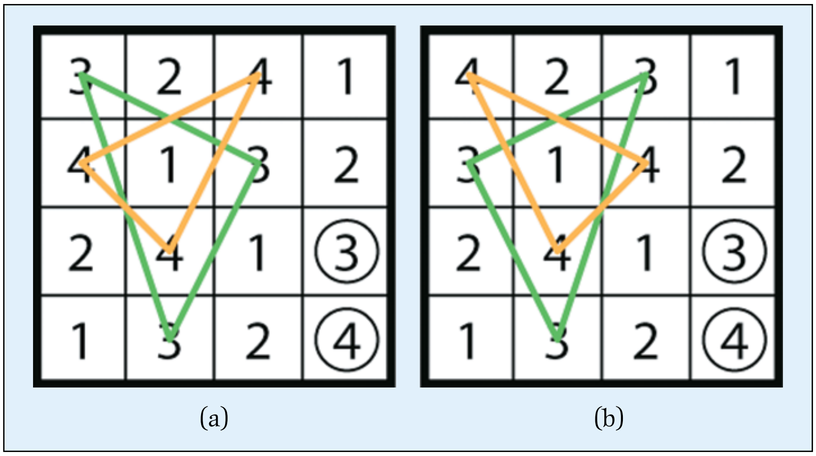
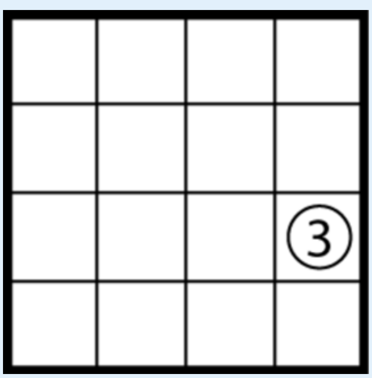


Fig. 12 Nate's original puzzle had two possible solutions.



1 Rhombus
2 Square
3 Isosceles right triangle
4 Quadrilateral

Fig. 13 Nate revised his 4 × 4 Shapedoku puzzle so that it has one solution.

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