Using KenKen to Build Reasoning Skills

Through KenKen puzzles, students can explore parity, counting, subsets, and various problem-solving strategies.

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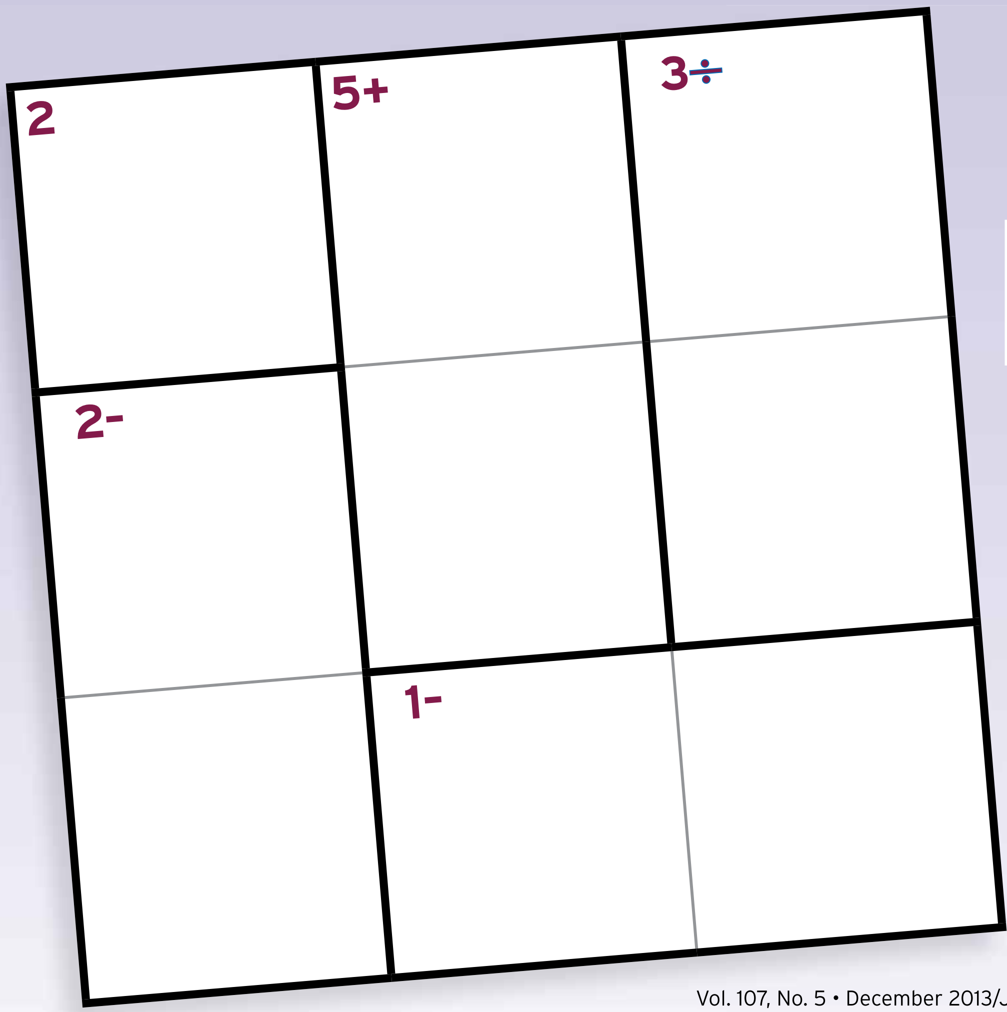

KenKen® is the new Sudoku. Like Sudoku, KenKen requires extensive use of logical reasoning. Unlike Sudoku, KenKen requires significant reasoning with numbers and operations and helps develop number sense.

The creator of KenKen puzzles, Tetsuya Miyamoto, believed that "if you give children good learning materials, they will think and learn and grow on their own" (Nextoy 2012). We agree. KenKen puzzles help develop perseverance and stamina, and in the classroom they promote problem solving, reasoning, and communication. Because KenKen puzzles can vary in difficulty, they can be used effectively in middle school classrooms, high school classrooms, extracurricular math clubs, mathematics courses for nonmajors, and methods courses for preservice teachers.

At their core, KenKen puzzles involve simple arithmetic, but solving them requires a combination of logic, algebra, number theory, and combinatorics. While solving the puzzles, students practice addition, subtraction, multiplication, and division; consider multiple factorizations and partitions of numbers; and invoke deductive reasoning. In a high school classroom, KenKen puzzles can also be used to develop algebraic thinking, explore syllogisms and isomorphism, investigate topics in discrete math, and reinforce geometry concepts.

Before reading further, you might try to solve a puzzle for yourself. We suggest the sample problem in **figure 1**. This is a 3×3 puzzle in which the numbers 1, 2, and 3 occur once in each row and column. The two numbers in the [1–] "cage" have a difference of 1, and the six numbers in the [10+] cage have a sum of 10. The notation [$n*$] is used to indicate a cage for which the operation $*$ is used to obtain a result of n. (For additional examples, visit http://illuminations.nctm.org/kenken, where four new puzzles appear daily.)

You may not want to read the following solution until you have attempted the puzzle on your own (this advice applies to all puzzles presented here):

- The [3] cage must be filled with a 3.
- The sum of all nine entries is $3 \times (1 + 2 + 3) = 18$, so the sum of the entries in the [1–] cage must be $18 - 3 - 10 = 5$. Consequently, the [1–] cage must be filled with 2 and 3, and their order is dictated by the 3 in the [3] cage.
- Once a 2 and two 3s are placed in the [3] and [1–] cages, the rest of the numbers fall into place. The solution is shown in **figure 2**.

In general, the rules for KenKen are as follows:

- For an $n \times n$ grid, fill each row and column with the numbers 1 through n. A number may not be repeated within any row or column.
- Each heavily outlined set of cells, called a cage, contains a mathematical clue that consists of a number and an arithmetic operation: +, –, ×, or ÷. The numbers in that cage must combine (in any order) to produce the target number using the mathematical operation indicated.
- Cages with just one cell should be filled with the target number.
- A number may be repeated within a cage, provided it is not in the same row or column.

We will demonstrate several mathematical reasoning strategies that can be used to solve KenKen puzzles and to highlight some benefits of using numerical puzzles in the classroom. In particular, KenKen puzzles allow students to explore basic operations, factors, parity, symmetry, modular arithmetic, congruence, isomorphism, and algebraic thinking. Perhaps more important, however, they allow students to engage in the mathematical practices identified in the Common Core standards. As we review strategies for solving KenKen puzzles, we will attempt to indicate opportunities where these strategies can be used with students.

STRATEGIES

The two requirements of a KenKen solution are that the digits 1 through n must be used in an $n \times n$ grid and that the mathematical clues must be satisfied. Within a puzzle, each cage may have just one set that fulfills the clue, or there may be several

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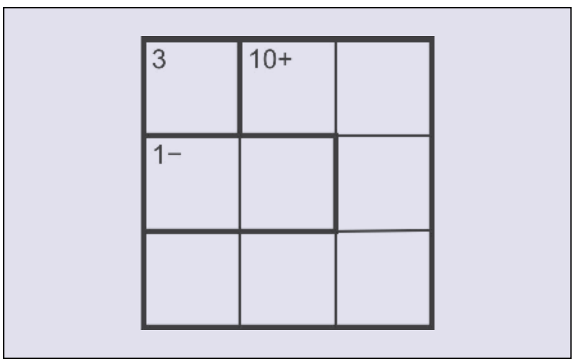

Fig. 1 The grid can be filled in only one way that satisfies all the requirements of this puzzle.

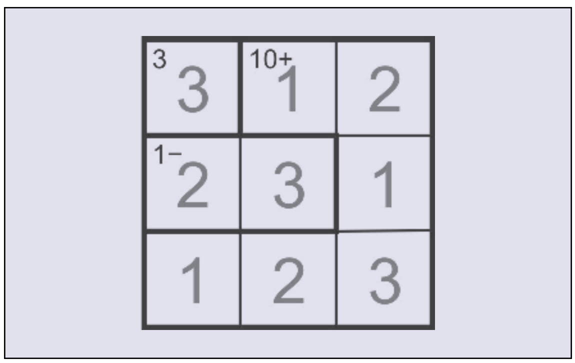

Fig. 2 This is the solution to the sample 3×3 KenKen puzzle that appears in **figure 1**.

candidate sets that could fill the cage. Our approach depends on the clue as well as the size of the puzzle. When appearing in a 6×6 puzzle, for example, the two-cell cage [7+] has three candidate sets—$\{1, 6\}$, $\{2, 5\}$, and $\{3, 4\}$. In a 4×4 puzzle, however, the same cage has a unique candidate set—$\{3, 4\}$.

This idea of candidate sets is important when considering strategies. The examples that follow highlight the importance of this idea.

The X-Wing Strategy

The X-wing strategy is borrowed from Sudoku. As the name implies, cells from multiple rows and cages are used in tandem to eliminate candidates. In essence, the X-wing strategy is a process of creating an organized list and eliminating possibilities, a problem-solving technique listed by Pólya in *How to Solve It* (1945).

Because k parallel lines cannot have more than k copies of a given number, we can eliminate candidate sets for the second and third rows of the 6×6 KenKen puzzle in **figure 3**.

Observe that the [15×] cage has only one possible candidate set: $\{1, 3, 5\}$. On the other hand, the [18×] cage has two possible candidate sets: $\{1, 3, 6\}$ and $\{2, 3, 3\}$. When $\{1, 3, 5\}$ is used to fill [15×], however, it would be impossible to fill [18×] with $\{2, 3, 3\}$ because three 3s would appear in these two rows—a violation of the rules. Consequently, [18×] must be filled with $\{1, 3, 6\}$.

Fault Lines and Parity

A fault line is a heavy horizontal or vertical line that cuts entirely through a puzzle, dividing it into two pieces. Pólya advised, “If you can’t solve a problem, then there is an easier problem you can solve: Find it” (1945, p. 114). Fault lines cut a puzzle into smaller rectangles of manageable size. The parity of a cage is even (odd) if the sum of the entries of the cage is even (odd). For example, [11+] is an odd cage because the sum of the entries is 11, which is an odd number. The parity of some two-cell cages is determined, even though the candidates are not. For example, [2–] is an even cage because the entries are either both even or both odd; in both cases, the sum of the entries is even. On the other hand, some two-cell cages can be either even or odd. For example, [12×] has two pairs of candidates; one is $\{2, 6\}$, which is even, and the other is $\{3, 4\}$, which is odd.

Parity can be used to make progress in the first row of the puzzle in **figure 3**. Because the sum of the six entries is $1 + 2 + 3 + 4 + 5 + 6 = 21$, an odd total, the row must have either one or three odd cages. Since the two [1–] cages are odd, the [12×] cage must also be odd. Therefore, the [12×] cage must be filled with $\{3, 4\}$.

Parity is a topic from number theory that can also be useful even when no fault line exists. Consider the second and third rows from the puzzle in **figure 3**.

Notice that three cages—[18×], [6+], and [12+]—are even cages, whereas [15×] is an odd cage. The sum of all entries in these two rows is $2 \times 21 = 42$.

That KenKen puzzles can be solved in multiple ways is one reason why they are so powerful in the classroom.

Therefore, parity logic dictates that the entry in the top cell of the [11+] cage must be odd, so it has to be filled with 5.

KenKen puzzles often have multiple solution paths. We have used the previous example to highlight the use of parity, but we do not mean to suggest that parity is the only viable method. For instance, it could be noted that the [20×] cage in the fourth row must be filled with $\{4, 5\}$, so the lower cell in the [11+] cage must be 6 and the top cell must be 5.

That KenKen puzzles can be solved in multiple ways is one reason why they are so powerful in the classroom. A discussion about various strategies allows students to “identify correspondences between different approaches” (CCSSI 2010), an idea emphasized in the Common Core standards.

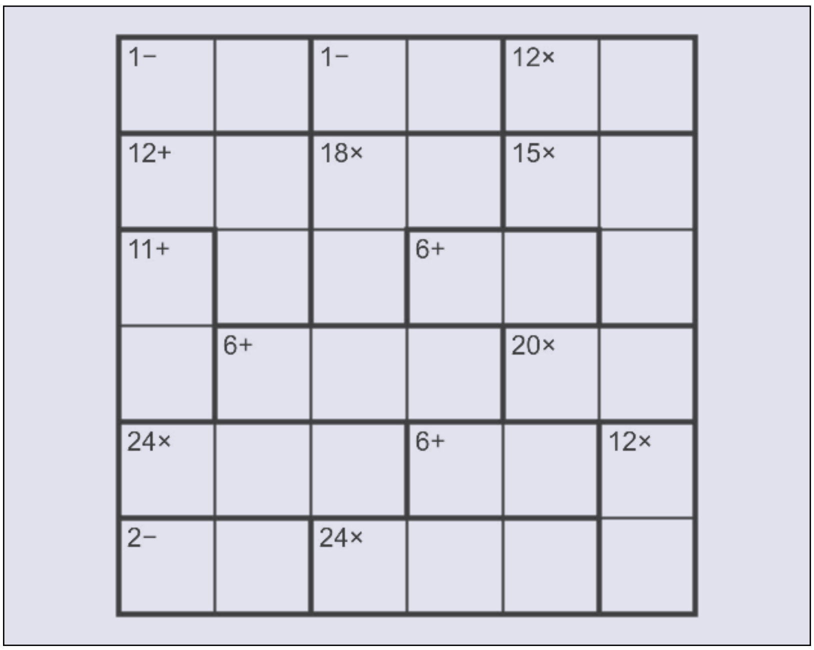

Fig. 3 This KenKen puzzle can be used to demonstrate a number of solution strategies.

Stacked Cages

Some puzzles have two or more cages confined to a single row or column. These stacked cages can be used to eliminate candidate sets.

In the classroom, students construct syllogisms by connecting pieces of information. Information about the stacked cages in the bottom row of the puzzle in **figure 3** can be used to exemplify the type of deductive reasoning in which students might engage while solving a KenKen puzzle:

In addition to allowing students to apply deductive logic, orthogonal and parallel cages also provide an opportunity to use geometric terms outside the regular curriculum.

- The candidate sets for [24×] are {1, 4, 6} and {2, 3, 4}.
- If the candidate sets are {1, 4, 6} and {2, 3, 4}, then [24×] must contain a 4.
- If [24×] contains a 4, then [2–] cannot contain a 4.
- If [2–] cannot contain a 4, then the only candidate sets are {1, 3} and {3, 5}.
- If the only candidate sets are {1, 3} and {3, 5}, then [2–] must contain a 3.
- If [2–] must contain a 3, then [24×] cannot contain a 3.
- If [24×] cannot contain a 3, then it must be filled with {1, 4, 6}.
- If [24×] is filled with {1, 4, 6}, then [2–] must be filled with {3, 5}.
- If [24×] is filled with {1, 4, 6} and [2–] is filled with {3, 5}, then the last cell in the sixth row must be filled with 2.

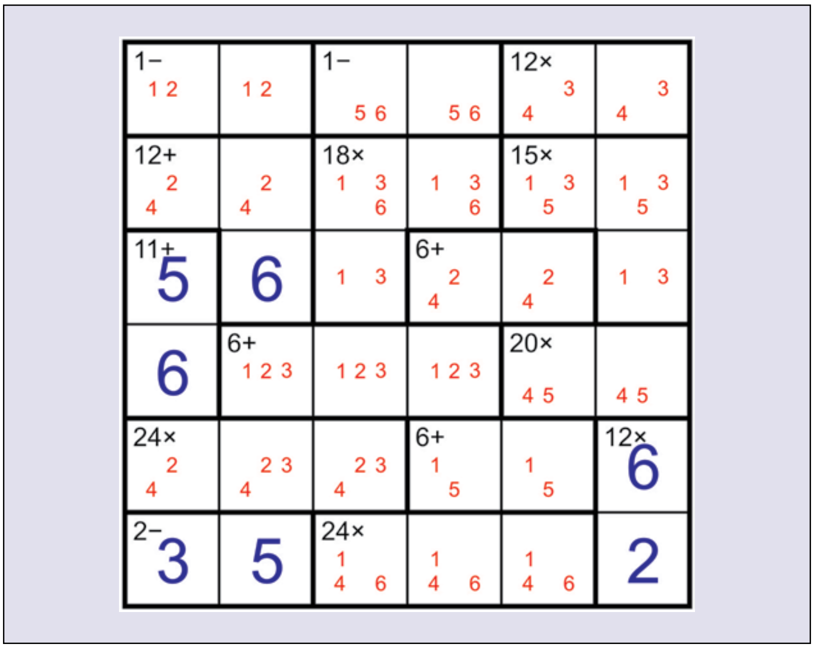

Fig. 4 The 6 × 6 KenKen puzzle that appears in **figure 3** has been partially solved. A blue numeral has been determined for the cell, whereas red numerals are possible candidates.

Parallel and Orthogonal Cages

Orthogonal cages are two [$n*$] cages oriented at right angles to each other. As a result, the two cages cannot be filled with the same set of numbers. The two [12×] cages that appear in the first row and the sixth column of the puzzle in **figure 3** are orthogonal cages. Because [12×] cages have only two possible candidate sets, {3, 4} and {2, 6}, one of those sets must appear in each [12×] cage. The results from the stacked cages strategy dictate that {2, 6} must fill the [12×] cage in the last column, so {3, 4} must be used for the [12×] cage in the top row.

Related to orthogonal cages are parallel cages, which are two-cell [$n*$] cages that appear in parallel lines in the same position within the line. The required uniqueness of the solution implies that the two cages cannot be filled with the same two-element set. If they were filled with the same set, it would be impossible to know the order of the numbers.

As an example, consider the third and fifth rows of the puzzle that appears in **figure 3**. Both rows contain a [6+] cage occupying the fourth and fifth cells. There are only two possible candidate sets: {1, 5} or {2, 4}. Therefore, we must use each of these sets in one of the [6+] cages. Further, the first cell of the third row contains a 5, so the [6+] cage in the third row must be filled with {2, 4}.

In addition to allowing students to apply deductive logic, orthogonal and parallel cages also provide an opportunity to use geometric terms outside the regular curriculum.

Elimination

We have already determined that the [12×] cage in the top row must be filled with {3, 4}. That leaves us with {1, 2, 5, 6} to fill the two [1–] cages in the top row. We can do so only if we use {1, 2} in one of the [1–] cages and {5, 6} in the other. Because {5, 6} is used to fill the [11+] cage in the first column, {1, 2} must be used to fill the first [1–] cage in the top row, and {5, 6} must be used to fill the second [1–] cage.

A similar but simpler process of elimination can be used in the fifth row. We know that the last cell in the fifth row must contain a 6 and that the [6+]

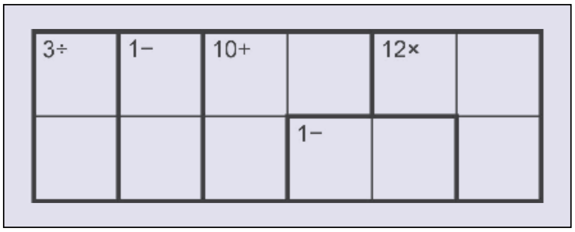

Fig. 5 Two isolated rows from a 6 × 6 puzzle illustrate a parity strategy.

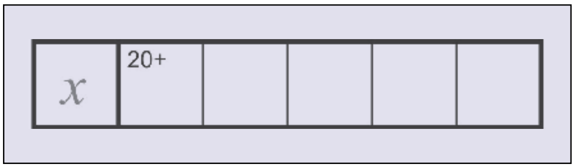

Fig. 6 An isolated row from a 6 × 6 puzzle provides an opportunity to apply a counting strategy.

cage must be filled with {1, 5}, so the [24×] cage can be filled only with {2, 3, 4}.

As a result of the strategies used thus far, the puzzle is now partially completed (see **fig. 4**). (A large numeral indicates that the number for that cell has been determined; several smaller numerals within a cell indicate candidates.) There is still work to be done, but the remainder of the reasoning is straightforward.

For instance, the [1–] cage and the [12+] cage in the upper-left corner of the puzzle must each contain a 2. Consequently, the X-wing strategy dictates that the first two cells of the [24×] cage in the fifth row cannot contain a 2; further, because the first cell of the [24×] cage cannot be a 3, it must be a 4. By elimination, the middle cell must be a 3.

Similar reasoning can be used to complete the puzzle. Good luck!

OTHER STRATEGIES

The 6 × 6 puzzle in **figure 3** was used to illustrate several strategies, but not all solution strategies can be shown in a single puzzle. Here we provide additional examples to highlight other strategies that may be useful when solving KenKen puzzles.

Advanced Parity

Parity is an integral topic in number theory, and advanced uses of parity can be helpful in solving KenKen puzzles.

Consider the two rows of a 6 × 6 KenKen puzzle shown in **figure 5**. The set for the [12×] cage must be one of {1, 3, 4}, {1, 2, 6}, or {2, 2, 3}. (This last set is possible because the cage is L shaped; the two 2s can occupy different rows and columns.) But parity can be used to exclude {1, 2, 6} and {2, 2, 3} because both are odd. The cages [3×] and [10+] are both even, and the two [1–] cages are both odd. Because the sum of the entries in the two rows must

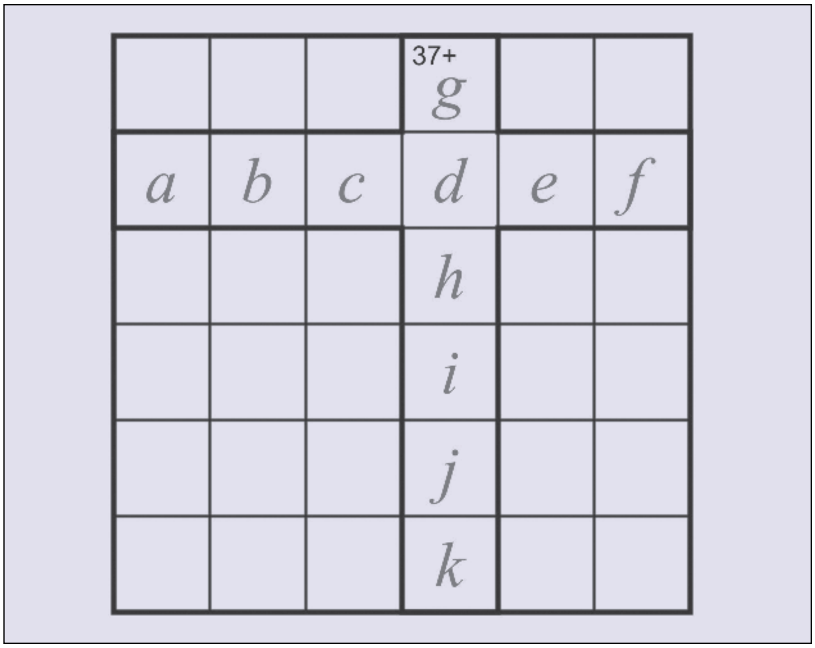

Fig. 7 Counting techniques can also be used in situations like this.

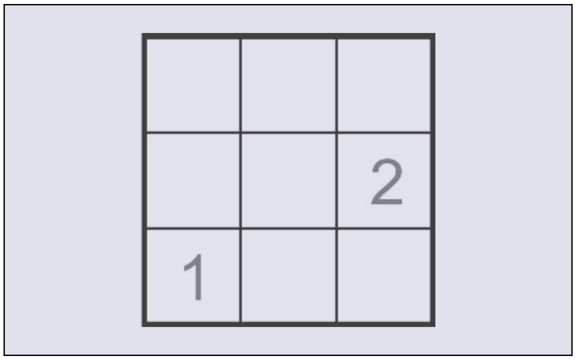

Fig. 8 The unique candidate rule can be used to complete this Latin square.

be 42, the number of odd cages must be even. Therefore, the [12×] cage must be even, so the only possible candidate set is {1, 3, 4}.

Parity refers specifically to numbers modulo 2, but a similar idea can be used with other bases. For an example using modulo 3, see puzzle 3 in the additional puzzles and solutions provided at **www.nctm.org/mt047**.

Counting

As the name implies, counting uses basic arithmetic facts to determine the number for a cell. However, because algebra is the generalization of arithmetic, KenKen puzzles provide an opportunity to promote algebraic thinking.

Figure 6 shows one row from a 6 × 6 puzzle. What is the value of x? Because the sum of the entries in the row must be 21, $x = 1$.

The example shown in **figure 7** demonstrates a more advanced use of the counting technique. The eleven entries in the [37+] cage have a sum of 37, but

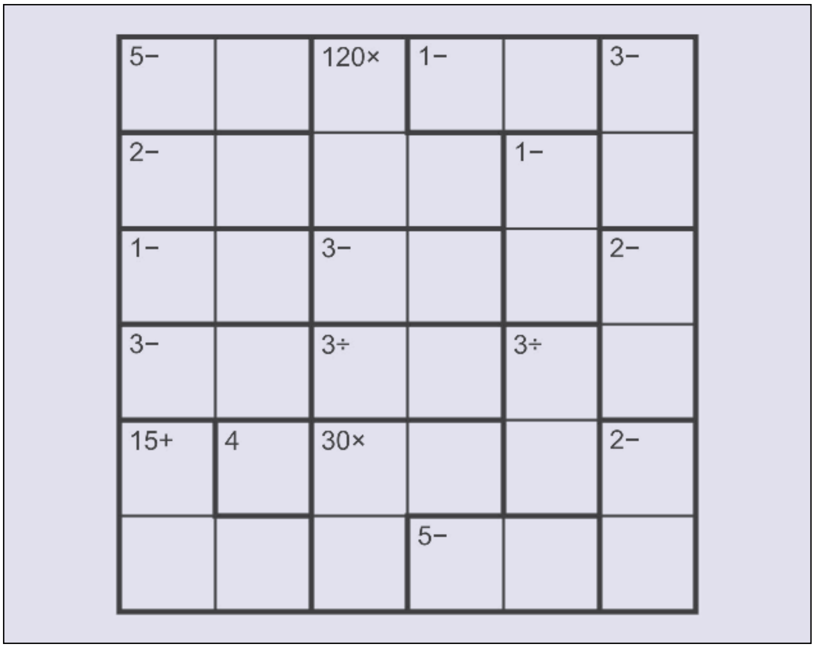

Fig. 9 This 6×6 KenKen puzzle contains two vertical fault lines.

the sum of the entries in the row is $a + b + c + d + e + f = 21$, and the sum of the entries in the column is $g + d + h + i + j + k = 21$, yielding the following:

$$\begin{aligned}(a + b + c + d + e + f) + (g + d + h + i + j + k) &= \\ (a + b + c + d + e + f + g + h + i + j + k) + d& \\ 21 + 21 &= 37 + d \\ 42 &= 37 + d\end{aligned}$$

Consequently, $d = 5$.

The Unique Candidate Rule

Once $n - 1$ copies of a digit have been placed, the location of the last copy is determined. The unique candidate rule can be exploited to complete the 3×3 Latin square shown in **figure 8**.

The lower-right corner must contain a 3 because a 1 already appears in the third row and a 2 already appears in the third column. Then the second cell of the third row must be a 2. At this point, the unique candidate rule can be invoked: Because both the second and third rows as well as the second and third columns contain a 2, the last remaining 2 must appear in the upper-left corner. Filling the remaining cells is straightforward.

The unique candidate rule is powerful for solving KenKen puzzles, especially when many cells have been filled. However, this rule is a general Sudoku strategy and not specific to KenKen. Moreover, all Sudoku strategies are relevant to KenKen because both types of puzzles rely on Latin squares.

Subset Analysis

The 6×6 puzzle shown in **figure 9** contains two vertical fault lines. Consider the cells in the third, fourth, and fifth columns. Two cages in these columns have just one candidate set. They are [120×], which has only $\{4, 5, 6\}$, and [5–], which has only $\{1, 6\}$. But notice that the two [3÷] cages are orthogonal, meaning that they must be different, so one is $\{1, 3\}$ and the other is $\{2, 6\}$. Thus, all three of the 6s in these three columns have been used, implying that the [30×] cage must be filled with $\{2, 3, 5\}$. Taken together, these five cages account for $\{1, 1, 2, 2, 3, 3, 4, 5, 5, 6, 6, 6\}$, leaving $\{1, 2, 3, 4, 4, 5\}$ to fill the remaining two-cell cages: [1–], [1–], and [3–].

At this point, we can use subset analysis. If $\{2, 5\}$ were used to fill the [3–] cage, there would be no way to match the 1 with another digit to fill a [1–] cage. Consequently, the [3–] cage must be filled with $\{1, 4\}$, and the two [1–] cages must be filled with $\{2, 3\}$ and $\{4, 5\}$.

VARIANTS

Some variations of KenKen puzzles use slightly modified rules. For instance, Turbo KenKen uses characters other than 1 through n, and Primal KenKen uses only prime numbers. In Abstract KenKen, the characters need not be numbers, and the operations need not be the four binary operations.

KENKEN AND MATHEMATICS

We have used KenKen puzzles successfully and in myriad ways with students from prekindergarten through college. As a class exercise, one author gave each student two copies of a different KenKen puzzle. Students solved the problem on one copy and then used the second copy to explain their solution to a partner. Through these discussions, a number of the strategies that we have discussed here were uncovered, and the other strategies were then shared with a class of highly engaged students.

Increased discourse is just one of the reasons to use KenKen puzzles in a mathematics classroom. They encourage problem solving, reasoning, and mathematical communication, and they also promote important mathematical practices, such as perseverance and the ability to examine the reasonableness of a result. In addition, students must reason quantitatively to solve a KenKen puzzle, but they must reason abstractly when creating a puzzle on their own. When one high school teacher had his students solve and then create KenKen puzzles, he found that the enthusiasm and results exceeded his expectations.

Arithmetic is at the heart of KenKen, and the concepts of factor and partition are ubiquitous. But more advanced topics can be explored as well. Modular arithmetic and congruence are standard topics in number theory; isomorphism and mapping functions can be used to show the relationship between two puzzles; parity and algebra can be used to make progress on a puzzle; and linking several results to form an extended syllogism is a form of deductive proof.

Not only have we had success reaching students using KenKen, but we have also had quite a bit of fun ourselves.

Do you KenKen? We think you should.

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