The practice of problem posing is as important to develop as problem solving. The resulting explorations can be mathematically rich.

A Rationale for Irrationals:

An Unintended Exploration of

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Although mathematicians from many early cultures assumed that all positive numbers could be written as a ratio of two natural numbers (i.e., rational numbers), irrational numbers are in fact the solution to many interesting problems in mathematics. The diagonal of a unit square and the circumference of a unit circle, for example, both have irrational lengths. Indeed, irrational numbers can arise from deceptively simple situations. What began as a mathematical task to encourage problem solving unexpectedly led to the irrational number *e*.

Today, the Common Core State Standards for Mathematics (CCSSI 2010) expect students in as early as eighth grade to be knowledgeable about irrational numbers. In particular, students are to understand infinite decimal expansions and convert between rational numbers' decimal expansions and fractional forms (8.NS.1) and arrive at increasingly better approximations for irrational numbers through iteration (8.NS.2). Later, they are required to learn how to operate with and understand properties of irrational numbers (N-RN.3).

Yet a common tendency in classrooms and on standardized tests is to avoid rational and irrational solutions to problems in favor of integer solutions, which are easier for students to comprehend and check. Without practice and without the mathematical sophistication needed for more rigorous proofs about the irrationality of numbers, students' appreciation of the importance and behavior of irrational numbers may be limited. Nonetheless, activities such as looking at iterations of squared numbers that get closer to 13 (Lewis 2007), constructing the number line (Coffey 2001), and studying the circumference of a circle through regular n-gons as n gets large (Wasserman and Arkan 2011) promote understanding of the infinite behavior of irrational numbers and help develop students' broader conceptions of number.

Common irrational numbers (such as π, square roots, e, and logarithms) as well as their applications in our world are frequently approached from their traditional developments. The irrational number e, for example, is often discussed in the familiar contexts of continuously compounded interest, which results from the fact that

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$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e;$$

or, in calculus, as the exponential function with the property that $f'(x) = f(x)$; or as the sum of an infinite series,

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e.$$

In contrast with these more traditional developments, this article presents an alternative approach to uncovering the irrational number e. The activity illustrates a "low entry, high ceiling" mathematical task, which could be used in a variety of mathematics courses to build appreciation for and understanding of irrational numbers. What started as a problem-solving activity with in-service K–12 teachers led surprisingly to the irrational solution, e. This article provides an exploration of how the irrational number e emerges from various solution strategies to an unconventional problem and an example of how problem posing can lead to interesting mathematics.

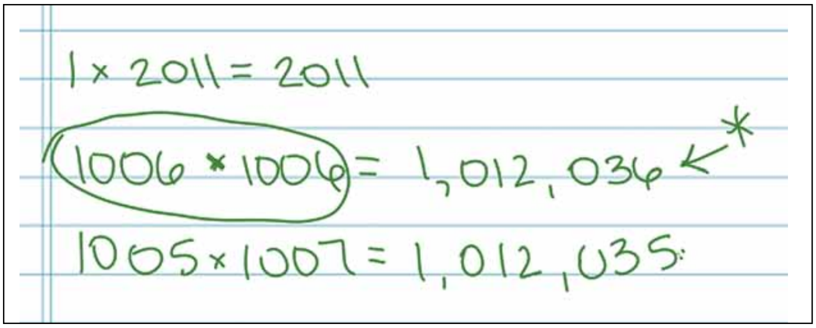

Fig. 1 Students easily found the maximum product for two numbers.

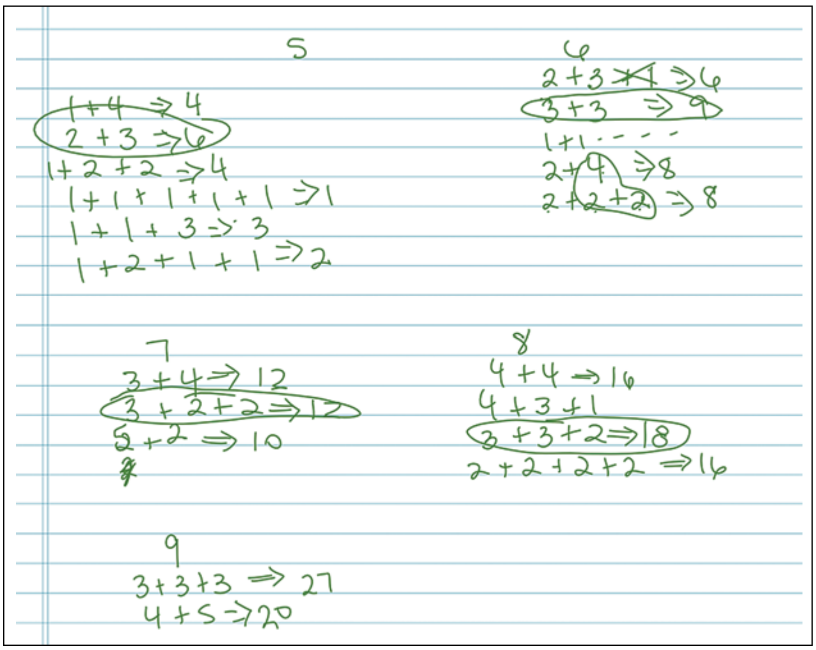

Fig. 2 Some students explored simpler problems.

The term *problem posing* is borrowed from *The Art of Problem Posing* (Brown and Walter 2005) and refers to shifting the mathematical involvement of students to include not just the solving but also the formulation of problems. In particular, Brown and Walter articulate the connection between problem solving and problem posing in mathematics and argue that this interaction potentially deepens understanding and fosters creative inquiry. The "What-If-Not" approach they describe for posing mathematical problems, of which this article provides one example, encourages students to list specific attributes of a problem and then explore the mathematical implications for modifying the problem by removing one or more assumptions.

PARTITIONING FOR MAXIMUM PRODUCT

In a graduate mathematics education course titled Numbers and Operations, in-service K–12 teachers (referred to here as students) were given the following problem-solving task:

> Find positive integers whose sum is 2012 and whose product is as large as possible.

A large number (2012 is arbitrary) requires students to use reasoning and problem-solving techniques rather than rely completely on calculators.

Students' first approach frequently involved splitting 2012 in half and comparing 1006×1006 with other possibilities, such as 1005×1007 (see **fig. 1**). When asked why 1006×1006 must be the maximum, students made comparisons to a square being the maximum-area rectangle with a given perimeter and to a vertex as the maximum of a parabola. Many students were satisfied with their answer; however, the instructor's probing about their assumption, which rests on splitting into two numbers, quickly prompted students to begin exploring partitions of more than two numbers: "Wait, so we could do multiple numbers, like 1000, two times, and 12 [1000 + 1000 + 12]." This partition gives a product $(1000 \times 1000 \times 12)$ that is approximately 12 times larger than 1006×1006. Realizing that the problem allows for such partitions inspired students to explore further. The two methods described here were developed during the course activity.

Method 1: Solve a Simpler Problem

Following one of Pólya's (1971) problem-solving strategies—"Solve a simpler problem"—some students began to explore the same problem but with smaller numbers. **Figure 2** reveals the work of students who looked at the maximum product for

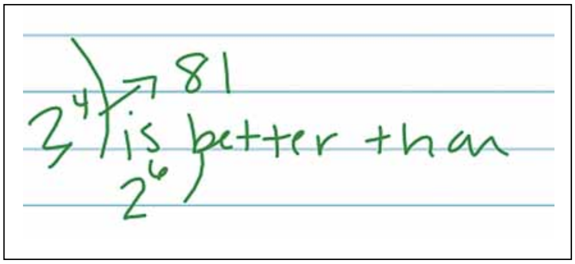

Fig. 3 Students reached a general conclusion about maximum product partitions.

partitions of 5, 6, 7, 8, and 9. (Students quickly realized that partitions using 1 did not increase the product.) Some groups "figured out it has to be combinations of 2s and 3s" but remained unsure about a more general conclusion. Their reasoning fluctuated from trying to have all of one or all of the other (i.e., all 2s or 3s) to having equal quantities of both. Not until students explored the partition of 12 did they generalize an important result (see **fig. 3**): Splitting into 3s provides a larger product because $2 + 2 + 2 + 2 + 2 + 2 = 3 + 3 + 3 + 3$, but $2^6 < 3^4$. Students verified that other examples, including 6, used mostly 3s as well.

The conclusion from these simpler problems was insightful: "So if we went back and broke [2012] into 6s. . . ." This thought helped some students overcome a common misconception in the original example of 2012—that the integer splits all had to be the same number (e.g., all 2s or all 6s). For any sum of 6, splitting into two 3s (and not three 2s) results in the maximum product; therefore, students reasoned that splitting 2012 into three hundred thirty-five 6s, with a remainder of 2, means that the maximum product must incorporate six hundred seventy 3s, with a 2 remaining—that is, the maximum product is $3^{670} \cdot 2$.

Method 2: Reason through Experimentation

Other students did not change the constraints of the original problem (they also used 2012), and, although the approach of exhausting all possibilities leads more toward exhaustion than a solution, a connection to the commutative property of multiplication led to some insight. Students observed that dividing 2012 by 4 could yield either four groups of 503 $(503 + 503 + 503 + 503)$ or five hundred three groups of 4 $(4 + 4 + 4 + \cdots)$. They realized that 4^{503} is significantly larger than 503^4 (see **fig. 4**).

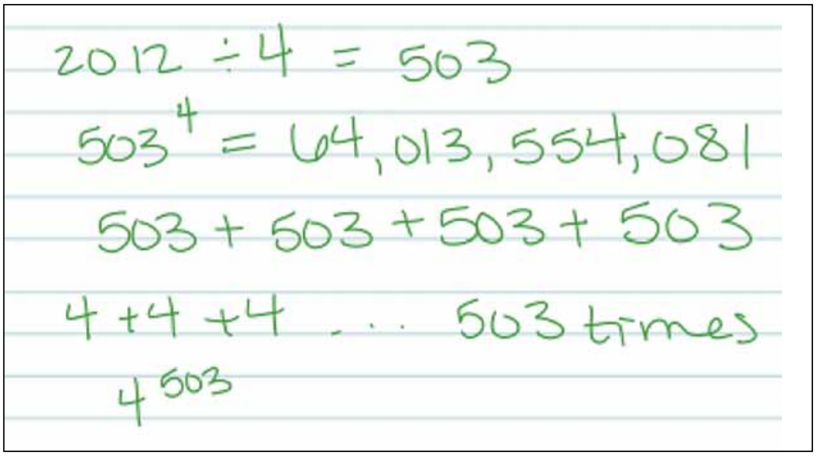

Fig. 4 Students gain insight into the partition.

Prompted by the instructor to generalize their conclusions thus far, students responded, "The more amounts we have to multiply by, the more numbers there are in the sum, the larger the [product]." This realization forced some students to conclude (falsely) that the largest exponent, which would come from splitting into groups of 2s, would be the largest product—that is, 2^{1006}.

One group connected this hypothesis with the fact that exponential functions grow more quickly than polynomial functions, implying that larger exponents would have larger products. This conclusion was not unreasonable, but the base of the exponent also matters; the instructor provided a counterexample from another group's work showing that $2^3 < 3^2$. These students then explored splitting 2012 into different groups but also initially had difficulty incorporating partitions that used different numbers (e.g., not all 2s or all 4s). Using 2012's prime factorization, they compared the six options (1^{2012}, 2^{1006}, 4^{503}, 503^4, 1006^2, and 2012^1) and concluded that both 2^{1006} and 4^{503} were sufficient because they are equivalent.

At this point, the instructor demonstrated, with an example of two hundred one 10s and one 2, how partitions could use different numbers and advised students to look at simpler problems, using an approach similar to method 1. These students likewise arrived at the conclusion that the maximum product partition would use groups of 3: $3^{670} \cdot 2$. (As an extension, students could use logarithms to compare this product with 2^{1006} on a calculator: $670 \cdot \log 3 + \log 2 > 1006 \cdot \log 2$.)

This second approach led to an intriguing question: Why is splitting into groups of 3 better than splitting into groups of 2 in this problem? (Or, in one student's words, "But are 3s better than 2s?") From the perspective of exponents, it was unclear why the smaller exponent on 3 in this problem yielded a bigger product than the larger exponent on 2. This question led to an unintended exploration of the irrational number *e*.

AN UNINTENDED EXPLORATION

A slight modification of the original task generated a new problem-solving adventure.

> Find positive *numbers* whose sum is 2012 and whose product is as large as possible.

One word, *integers*, was replaced; this small change removed an assumption about the original problem (that the partition must be an integer partition) and required approaching the task from a different perspective. Essentially, the new question can be read as, "*What if* the partitions did *not* have to be whole numbers?" Although it would not be a common alteration for discussing integer partitions in number theory, this "What-If-Not" modification fostered an opportunity to explore this problem from a new vantage point that led to deeper discoveries. Other educators (e.g., Whitin 2004) also have shared reports of mathematical opportunities provided by explorations that incorporate problem posing and students' observations.

The group of students who previously used method 2 realized that their examples produced a general pattern. Because they had begun by choosing to divide 2012 into groups of equal size—say, 4—the product in consideration was $4^{2012/4}$; similarly, for other examples, they considered $2^{2012/2}$ or $6^{2012/6}$. Repeated examples helped students notice that they were exploring products of the form $a^{2012/a}$. With some insight from algebraic manipulation (see **fig. 5**), the task of finding a maximum product was greatly reduced. Students needed to explore a single function.

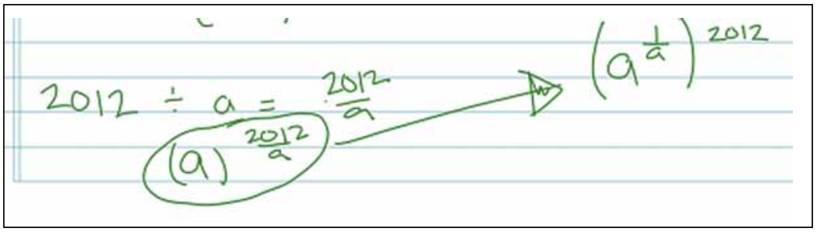

Fig. 5 Algebra directs students to the best base.

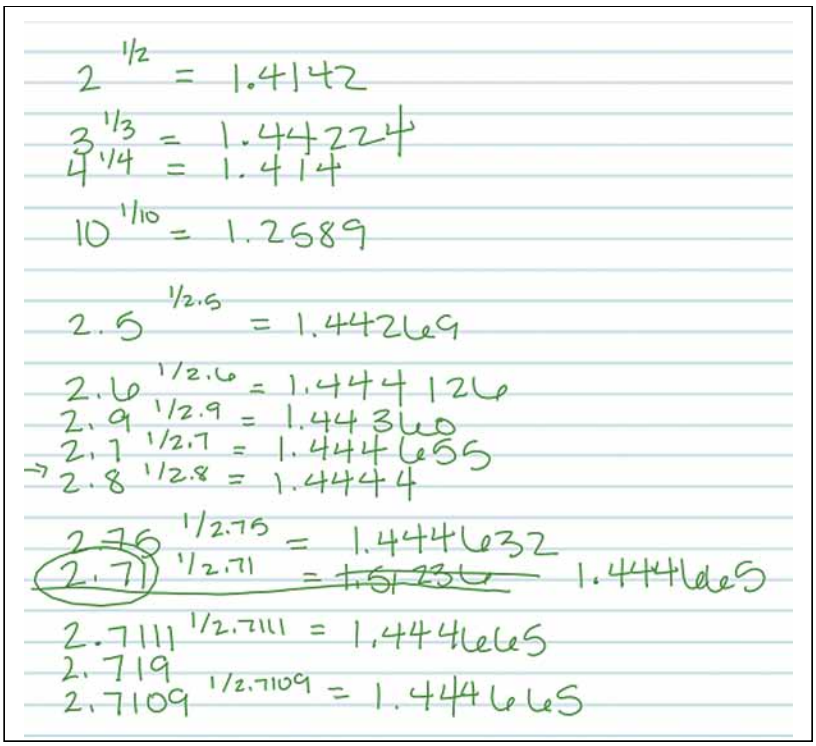

Fig. 6 Iterating to approximate the maximum of $f(x) = x^{1/x}$ points to e.

Although the implications were not immediately obvious, students focused on the function inside the parentheses: $f(x) = x^{1/x}$. Groups reasoned that this function had to be as large as possible and that the constant exponent of 2012 would not affect the maximum partition. An exchange between students illustrates this point:

Student 1**:** Will putting in 2012 change [the maximum]?

Student 2**:** Well, no . . . we're just looking to figure out what the x is. We would just do this part [i.e., $f(x) = \sqrt[x]{x}$], and then the 2012 will be maximized too.

Student 1**:** Oh, yeah, it won't change [the answer].

At this point, students went down one of two paths, depending on their familiarity with calculus. Students less familiar with calculus recognized their task as essentially trying to find the largest value by guess and check, using calculations to get increasingly better approximations through an iterative process. For example, after reasoning that the maximum split should be between 2 and 3, one group started with $2.5^{1/2.5}$, which these students found to be larger than either $2^{1/2}$ or $3^{1/3}$. They continued, identifying that $2.6^{1/2.6}$ was even larger but that the result of $2.9^{1/2.9}$ went back down, providing a narrower range ($2.6 < x < 2.9$) for their search. This group continued in a semi-iterative fashion with increasing decimal accuracy, trying, among other possibilities, $2.75^{1/2.75}$ and $2.7111^{1/2.7111}$.

As is evident from **figure 6**, these students were able to justify that 2.71—and indeed, 2.718—would result in a larger product. As students' interest waned in finding increasingly better estimates, the group was prompted to discuss the potentially infinite nature of the decimal expansion and if or when it might end. Students recognized limitations of the calculator for distinguishing better estimates (at some point, their efforts to maximize were indistinguishable on the calculator) but felt that the process for improving the estimates would continue indefinitely; a discussion comparing rational and irrational decimal expansions followed. Although the work from this approach cannot conclude with certainty that the best partition would be irrational, it did force students (in this case, primarily K–8 in-service teachers) to consider noninteger numbers as potential solutions to mathematical problems and reinforced differences between rational and irrational decimal expansions.

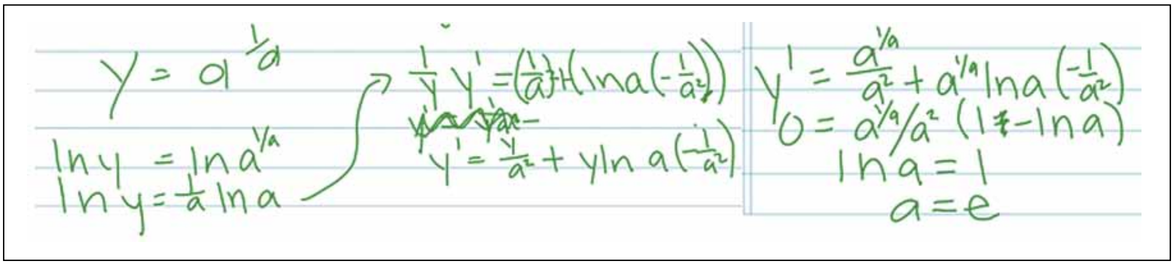

Fig. 7 Logarithmic differentiation is used to locate the maximum.

Students more familiar with calculus, for whom the term "maximize a function" prompted solving for a value where the derivative function is zero, used a different path for evaluation. Although their initial attempts misused the power rule for polynomials, they eventually found the derivative of $f(x) = x^{1/x}$ by using logarithmic differentiation (see **fig. 7**) to obtain

$$f'(x) = x^{\frac{1}{x}} \cdot \frac{1}{x^2}(1 - \ln x),$$

which they set equal to 0 and solved:

$$\begin{aligned} f'(x) &= x^{\frac{1}{x}} \cdot \frac{1}{x^2}(1 - \ln x) = 0 \\ x^{\frac{1}{x}} &\cdot (1 - \ln x) = 0 \\ 1 &= \ln x \\ x &= e \end{aligned}$$

The result demonstrates that e gives the precise maximum of this function (one student shouted exuberantly, "Oh, $x = e$! It's e! Like 2.7, e!"), which echoes the approximations from the previous group and can be shown graphically as well (see **fig. 8**).

This outcome helps answer two current questions: What is the partition of a number into positive numbers (not necessarily positive whole numbers) that yields a maximum product? And, returning to the original problem, why is splitting into groups of 3 better than splitting into groups of 2? In particular, the best way to partition a number into positive numbers so that the product is a maximum is into groups of e. Theoretically, the maximum would be $e^{2012/e}$. Since $3^{1/3} > 2^{1/2}$, this analysis also helps explain why splitting into groups of 3 resulted in a larger product than splitting into groups of 2. However, because the irrational exponent in $e^{2012/e}$ does not represent a product in the way that e^3 represents $e \cdot e \cdot e$, the complete answer to the modified question about 2012 requires finding a rational approximation for e. Although time constraints limited any extended discourse on this topic, discussing the exact rational solution would be enlightening. Because $2012/e \approx 740.17$, there are two rational approximations of e to consider: $2012/740 \approx 2.7189$ and $2012/741 \approx 2.7152$. The sum of 740 groups of 2012/740 and 741 groups of 2012/741 are both 2012; computing their products verifies that $(2012/740)^{740} > (2012/741)^{741}$.

The maximum product (requiring an integer exponent), then, is achieved at $(2012/740)^{740}$, where $2012/740 \approx 2.7189$ is the rational approximation of e that provides an exact solution. The theoretical maximum tends toward the irrational number e, although specific constraints may require a rational approximation. The precise solution to this problem could be used to promote further classroom discussion about irrational numbers and rational approximations for irrational numbers, a difficult concept to grasp (Arcavi, Bruckheimer, and Ben-Zvi 1987).

EXTENSION

Although this second "What-If-Not" problem has been discussed in relation to the ideas presented in method 2, the line of reasoning used in method 1 can also be generalized. In method 1, students reasoned about finding the maximum product partition with smaller numbers—for example, 12,

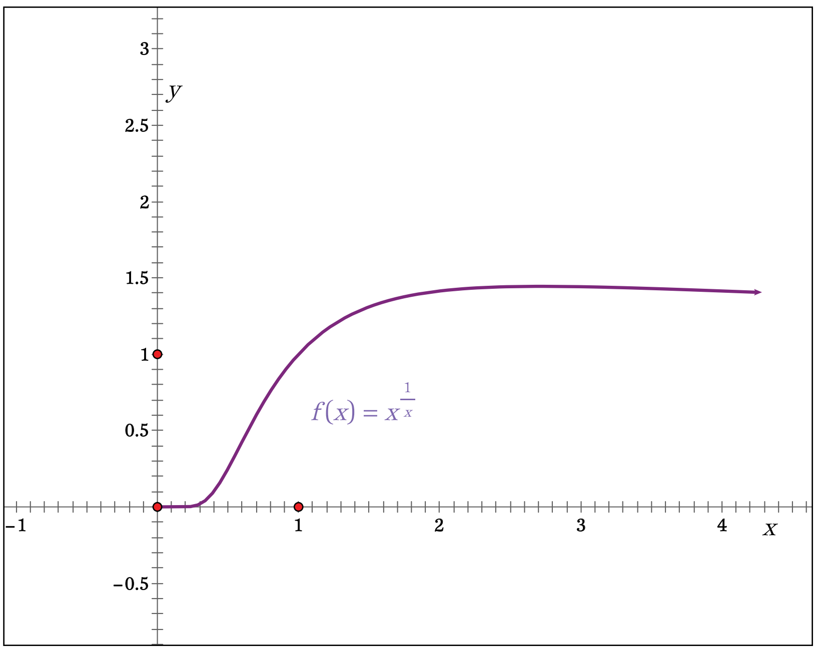

Fig. 8 Graphical representation shows that the maximum occurs at the irrational number e.

because $12 = 2 + 2 + 2 + 2 + 2 + 2 = 3 + 3 + 3 + 3$ but $2^6 < 3^4$. A composite number, expressed as the product of two factors, x and y, can be split not only into y xs but also into x ys. This observation is equivalent to recognizing that, for the given number,

$$\underbrace{x + x + x + \cdots}_{y} = \underbrace{y + y + y + \cdots}_{x}.$$

Because both sides sum to the same number, the question becomes, Which is bigger—x^y or y^x? Assuming that $x^y < y^x$ (and knowing that both factors are positive) produces these equivalent statements:

- $\ln x^y < \ln y^x$
- $y \ln x < x \ln y$
- $\ln x/x < \ln y/y$

The two expressions in the last statement are of same form, and so the solution amounts to finding the maximum of another function: $f(x) = \ln x/x$. Maximization through graphical representations (see **fig. 9**) or calculus techniques ($f'(x) = (1 - \ln x)/x^2$) reiterates that e is the maximum product partition. (Indeed, the two functions, $f(x) = x^{1/x}$ and $f(x) = \ln x/x$, are meaningfully related.)

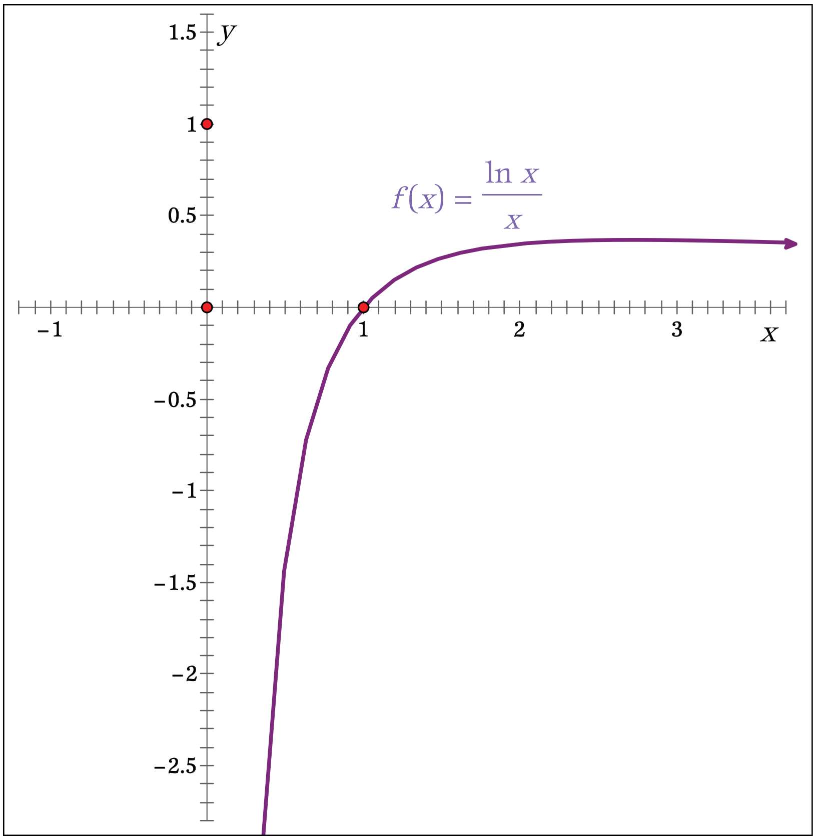

Fig. 9 A second graphical representation shows the maximum at $x = e$.

A DEEPER UNDERSTANDING

The original task—finding positive integers that sum to 2012 and whose product is as large as possible—was intended as an activity for in-service mathematics teachers. What started out as an exercise in problem solving turned into an opportunity to investigate the effect of problem posing; the consequences, notably, resulted in a much deeper discussion about numbers. Although partitions frequently refer to natural number partitions, letting go of that assumption and using the "What-If-Not" strategy for posing a modified problem (Brown and Walter 2005) fostered deeper insight into the original problem. The process and explorations forced students to refine the task (namely, that the task required finding the best base and not the largest exponent) and then to grapple with noninteger solutions. Consequently, students gained a deeper appreciation for the infinite nature of irrational numbers, generally, and properties of e, specifically. This activity typifies a "low entry, high ceiling" mathematical task, which can be a particularly effective means of engaging students in significant mathematical explorations.

Irrational numbers are the solution to many interesting problems in mathematics; connecting to students' interest in the mathematics of finance is a practical and useful way to explore the irrational number e. The maximum product partition problem, which requires knowledge of only basic operations (i.e., sum, product, exponents), provides an alternative investigation that is accessible to younger students. The property of e encountered in this problem can be explored before the notion of limit is necessary. Although precise justification of e as the solution may be difficult with prealgebra or algebra students, calculators can be used to obtain better estimates of the maximum through crunching numbers or graphing to explore maximums of a function; both are manageable investigations for earlier ages. Indeed, the activity (perhaps with a number smaller than 2012) could motivate students to explore the growth of exponential functions, the meaning of fractional exponents, applications of logarithmic properties, or iterating between increasingly better rational approximations of irrational numbers. For calculus students, the problem provides a context for logarithmic differentiation and solving logarithmic equations.

In mathematics education, problems are often designed to have integer solutions. However, including and discussing interesting problems with solutions that are not integers may develop in mathematics students (and teachers) an appreciation for the infinite decimal expansions of irrational numbers and an understanding of their practical—and rational—place in a world of irrationals.

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Editor's note: For more on partitioning, see the *MT* Calendar problem for May 13, 2013, which appeared in the April 2013 issue of the journal.

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