INVESTIGATING HOME PRIMES AND THEIR FAMILI

The process of prime factor splicing to generate home primes raises opportunity for conjecture and exploration.

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Since ancient Greek times, the study of prime numbers has fascinated generations of mathematicians, including Pythagoras, Euclid, and Eratosthenes. In the more modern era, contributors have included Thabit ibn Qurra, Girard, Fermat, Mersenne, Goldbach, Leibniz, Euler, Stern, Dirichlet, Cataldi, Fibonacci, Lucas, Legendre, Gauss, Bertrand, Chebyshev, Riemann, and Hadamard. Work done by these mathematicians has led to many types of primes (search "prime" at mathworld.wolfram.com), as well as algorithms and theorems attributed to them.

The notion of *home primes* is relatively new in the chronicle of mathematics. Heleen (1996–97) first described a procedure called *prime factor splicing*, hereafter referred to as PFS. The exploration of home primes is interesting and accessible to anyone who understands prime factorization.

To begin, let's consider any positive integer and resolve this integer into its prime factorization. Then we concatenate the prime factors in order of increasing or equal magnitude to form a new integer. If the new integer is composite, we repeat the first two steps. For example: $15 = 3 \cdot 5 \rightarrow 35 = 5 \cdot 7 \rightarrow 57 = 3 \cdot 19 \rightarrow 319 = 11 \cdot 29 \rightarrow 1129$ (prime), and $24 = 2^3 \cdot 3 = 2 \cdot 2 \cdot 2 \cdot 3 \rightarrow 2223 = 3 \cdot 3 \cdot 13 \cdot 19 \rightarrow 331319$ (prime).

The prime number eventually obtained in each of these two examples is known as the *home prime* of the original number. That is, 1129 is the home prime of 15, and 331319 is the home prime of 24. We will consider 1129 as the "parent" of the "child" 15, and, similarly, 331319 as the "parent" of the "child" 24. Note that the number 15 requires four iterations of

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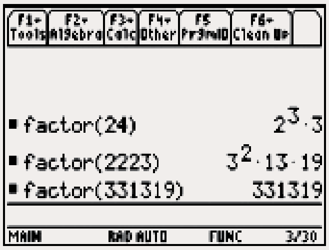

(a) TI-89

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factor(24)            2³ • 3
factor(2223)     3² • 13 • 19
factor(331319)         331319
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(b) TI-Nspire CAS

Mathematica reports
{prime factor, exponent of the factor}

In[1]= **FactorInteger[24]**
Out[1]= {{2, 3}, {3, 1}}
In[2]= **FactorInteger[2223]**
Out[2]= {{3, 2}, {13, 1}, {19, 1}}
In[3]= **FactorInteger[331319]**
Out[3]= {{331319, 1}}

(c) Mathematica

Fig. 1 CAS steps generate prime factorization needed for splicing.

the PFS process to reach its home prime parent and that the number 24 requires two iterations.

Computer algebra system (CAS) packages facilitate more extensive investigations than hand calculations and promote the exploration of many conjectures. **Figure 1** illustrates the steps needed to reach the parent of 24 using three different CAS technologies. The fact that factoring 331319 yields itself tells us that 331319 is prime. It would take considerably more time to determine by hand that 331319 is prime.

Heleen's algorithm for PFS initiated fascination with the topic of home primes and served as the impetus for mathematicians around the globe to delve more deeply into computation methods and conjectures. CAS technologies are now used to explore new mathematical insights and content. Some ongoing efforts to compute home primes and explorations of home prime properties are available at the World of Numbers recreational mathematics online forum (worldofnumbers.com/topic1.htm).

Investigating home primes can be a valuable experience at the secondary school level, a creative activity that fires the imagination of students as it introduces them to the research process. We have introduced the notion of home primes as part of a number theory unit in an effort to show students that mathematics is an ever-growing field of study, with new topics and themes continually emerging and developing, rather than being a stagnant field of facts. Although our students were preservice teachers, the following ideas can be adapted to students at many different levels in various classroom settings.

Students begin by finding all primes between 1 and 100 using the sieve of Eratosthenes. From there, we typically investigate applications of prime factorization, such as determining the number of factors for a given composite number, finding the greatest common factor and least common multiple of a given set of numbers, and establishing whether or not the decimal representation of a given fraction terminates or repeats. We then introduce home primes by demonstrating the PFS process. We challenge students to come up with ideas that could be considered for further study, keeping the focus on the set of positive integers. Through class discussion, we frame students' thoughts as questions such as these:

- If we start with a given prime number child, what is its parent?
- Does every composite child have a parent? If so, can we find the parent using our available CAS technologies?
- How many PFS iterations are needed to secure the parent of a given composite child? Moreover, is there a pattern to the number of iterations needed to find parents?
- Is it possible for more than one composite child to have the same parent?
- Is every prime number the parent to some composite child? If not, what characteristics make a prime number a parent?

The students then develop strategies for finding answers to their questions. In response to some of their questions, they recognize the need for clear mathematical definitions. We know that the only factor of 1 is itself, which is not a prime, so it can be argued that we cannot prime-splice 1. We also know that the only factors of a prime are that number and 1, so a prime is its own home prime. All prime numbers are considered to reach their home prime in zero iterations. We quickly find that starting with a composite integer is more interesting.

CONJECTURES ON COMPOSITE CHILDREN

Students have found building tables useful when pursuing their questions. As a homework assignment, they work on finding the parents for the seventy-four composite integer children between 1 and 100, inclusive. They are encouraged to use any CAS in their calculations and may later compare their results with those available at the World of Numbers forum. **Table 1** summarizes the home prime results.

Some students have become disappointed when trying to determine the parents for the children 49 and 77. One issue that arises is that the PFS process continues for many iterations. Second, even the most sophisticated technology fails to completely

Table 1 Home Prime Parents for Composite Children, 1 to 100

Integer	Number of Iterations	Home Prime		Integer	Number of Iterations	Home Prime
4	2	211		55	2	773
6	1	23		56	3	37,463
8	13	3,331,113,965,338,635,107		57	2	1129
9	2	311		58	1	229
10	4	773		60	2	35,149
12	1	223		62	3	31,237
14	5	13,367		63	1	337
15	4	1129		64	6	1,272,505,013,723
16	4	31,636,373		65	19	1,381,321,118,321,175,157,763,339,900,357,651
18	1	233		66	1	2311
20	15	3,318,308,475,676,071,413		68	2	3739
21	1	37		69	3	33,191
22	1	211		70	1	257
24	2	331,319		72	3	1,119,179
25	3	773		74	2	379
26	4	3251		75	2	571
27	4	13,367		76	3	333,271
28	1	227		77	unknown	still composite at 109th iteration
30	2	547		78	8	3,129,706,267
32	2	241,271		80	31	313,169,138,727,147,145,210,044,974,146,858,220,729,781,791,489
33	1	311		81	9	193,089,459,713,411
34	5	31,397		82	1	241
35	3	1129		84	1	2237
36	2	71,129		85	3	3137
38	2	373		86	17	6,012,903,280,474,189,529,884,459
39	1	313		87	18	41,431,881,512,748,629,379,008,933
40	9	3,314,192,745,739		88	2	719,167
42	2	379		90	3	71,171
44	9	22,815,088,913		91	7	236,122,171
45	6	3,411,949		92	2	331,319
46	1	223		93	1	331
48	15	6,161,791,591,356,884,791,277		94	2	1319
49	unknown	still composite at 110th iteration		95	4	36,389
50	2	3517		96	28	172,929,671,097,972,226,356,946,608,292,031,596,899,264,419
51	1	317		98	1	277
52	1	2213		99	2	71,143
54	1	2333		100	3	317,047

Table 2 Occurrences of Composite Children as a Function of PFS Iterations

Number of Iterations	Frequency of Occurrences
unknown	2
1	20
2	19
3	10
4	6
5	2
6	2
7	1
8	1
9	3
13	1
15	2
17	1
18	1
19	1
28	1
31	1

factor the very large integers that appear. To ease students' frustration, we inform them that some mathematical problems remain open despite the best efforts of even the most world-renowned mathematicians. We have found that the ability of current CAS technology to complete the factorization of a large composite number is contingent on the second largest prime factor. When this prime factor has more than, say, thirty digits, the factorization process can become stalled. (In computer science parlance, the algorithm used to generate the factors of a composite integer cannot be achieved in polynomial time and is termed NP Hard.)

Following their **table 1** homework, students build **table 2** in class, counting the number ("frequency of occurrences") of composite children requiring a given number of iterations to reach their parents. For example, three composite children (i.e., 40, 44, and 81) require nine iterations. Through this organization of data, students have discovered that only one or two iterations are needed to determine the parent of slightly more than half the composite children under consideration; further, of the known number of iterations, 31 is the maximum.

The numbers 49 and 77 have caught the curiosity of our students for another reason. Some

Table 3 Siblings

Set of Composite Integers	Common Home Prime
{49, 77}	unknown
{4, 22}	211
{12, 46}	223
{9, 33}	311
{42, 74}	379
{10, 25, 55}	773
{15, 35, 57}	1129
{14, 27}	13,367
{24, 92}	331,319

notice that in the first iteration of PFS, we have $49 = 7 \cdot 7 \rightarrow 77$, leading to the same steps and eventually (if it exists) the same prime as for 77. Thus, the children 49 and 77 must have the same parent, even though this home prime is currently unknown by the mathematical community. A check of **table 1** indicates that it is possible for other sets of composite children to share a common parent. Such numbers can be classified as "siblings" belonging to the same home prime "family." This finding inspires students to create **table 3**, showing sets of children from **table 1** that have a common parent. Note that **table 3** does not provide an exhaustive list of children for each parent because we limited our original set of composite children to natural numbers less than or equal to 100. For example, although two children—namely, 42 and 74—are listed for the parent 379, the composite number 237 is another child ($237 = 3 \cdot 79 \rightarrow 379$).

As a result of developing these three tables, students have wondered whether we can find other families of two or three siblings and whether we could find larger families with four or five siblings. These problems are open-ended with respect to the possibility of multiple correct answers. In each case, students would be expected to show PFS work to prove that the numbers they provide indeed have the same parent. Students could also be asked to explain their strategies for finding their solutions. One strategy is to examine a chain of iterations and flesh out the iterated composite children along the way to the parent.

CONJECTURES ON PARENTS

In addition to these specific problems relating to children, students have also come up with questions concerning parents. For example, they have inquired whether any composite child could have the parent 7, 17, 23, 37, 47, or 53. These problems are directly related to the following questions: Is every prime number the parent to some composite child? If not, what characteristics make a prime number a parent? Students have also queried whether the parent 773 can have children other than 10, 25, and 55. This question leads to a further one: How do we secure a child from any given parent? We have been surprised and impressed by our students' formation of problems of this type because addressing these questions requires reversing the PFS process. In the following examples, the direction of the arrows is reversed to show the backward process from the parent to the child.

Our students reason their way through the following conjectures: Single-digit primes (2, 3, 5, 7) cannot come from a concatenation procedure and thus cannot be considered parents. Similarly, two-digit teen primes (11, 13, 17, 19) do not come from a concatenation of two primes (the digit 1 is not prime) and thus cannot be considered parents. On the other hand, by PFS definition, two-digit prime numbers are parents if and only if the two individual digits are primes in ascending order. For example, the prime number 23 is the parent to the child 6. If we work backward, 23 would have had to have been found from concatenating the primes 2 and 3, whose product is 6 ($23 \leftarrow 2 \cdot 3 = \mathbf{6}$). Similarly, the prime 37 is the parent to the child 21 ($37 \leftarrow 3 \cdot 7 = \mathbf{21}$). The prime 47 cannot be a parent because the digit 4 is not prime. The prime 53 cannot be a parent because, although its digits are prime, its digits are not in ascending order.

As instructors, we help our students formalize their logical deductions. Suppose that our two-digit primes take on the form *ab*. Then we can express our current conjecture this way: *ab* will be the parent of a child $a \cdot b$ if and only if a and b are prime and $a \le b$. We encourage students to extend this thinking to larger-digit numbers, and we allow them to work in groups to figure out when three- and four-digit primes will work as parents. Strong reasoning skills must be used in this process, and many of our students do not find it easy to sort out the following patterns. However, we believe that, in the end, their struggles are rewarded with stronger logical thinking skills.

Three-digit primes work as parents if and only if the three individual digits are primes in increasing order (e.g., $227 \leftarrow 2 \cdot 2 \cdot 7 = \mathbf{28}$; $233 \leftarrow 2 \cdot 3 \cdot 3 = \mathbf{18}$; $257 \leftarrow 2 \cdot 5 \cdot 7 = \mathbf{70}$) or if the digits can be split left to right into a single-digit prime and a double-digit prime (e.g., $229 \leftarrow 2 \cdot 29 = \mathbf{58}$; $241 \leftarrow 2 \cdot 41 = \mathbf{82}$). For three-digit primes of the form *abc*, these two conditions can be expressed as

Table 4 Parent Sieve of First 550 Prime Numbers

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
53	59	61	67	71	73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173	179	181	191	193	197
199	211	223	227	229	233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349	353	359	367	373	379
383	389	397	401	409	419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541	547	557	563	569	571
577	587	593	599	601	607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733	739	743	751	757	761
769	773	787	797	809	811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941	947	953	967	971	977
983	991	997	1009	1013	1019	1021	1031	1033	1039	1049	1051	1061	1063	1069
1087	1091	1093	1097	1103	1109	1117	1123	1129	1151	1153	1163	1171	1181	1187
1193	1201	1213	1217	1223	1229	1231	1237	1249	1259	1277	1279	1283	1289	1291
1297	1301	1303	1307	1319	1321	1327	1361	1367	1373	1381	1399	1409	1423	1427
1429	1433	1439	1447	1451	1453	1459	1471	1481	1483	1487	1489	1493	1499	1511
1523	1531	1543	1549	1553	1559	1567	1571	1579	1583	1597	1601	1607	1609	1613
1619	1621	1627	1637	1657	1663	1667	1669	1693	1697	1699	1709	1721	1723	1733
1741	1747	1753	1759	1777	1783	1787	1789	1801	1811	1823	1831	1847	1861	1867
1871	1873	1877	1879	1889	1901	1907	1913	1931	1933	1949	1951	1973	1979	1987
1993	1997	1999	2003	2011	2017	2027	2029	2039	2053	2063	2069	2081	2083	2087
2089	2099	2111	2113	2129	2131	2137	2141	2143	2153	2161	2179	2203	2207	2213
2221	2237	2239	2243	2251	2267	2269	2273	2281	2287	2293	2297	2309	2311	2333
2339	2341	2347	2351	2357	2371	2377	2381	2383	2389	2393	2399	2411	2417	2423
2437	2441	2447	2459	2467	2473	2477	2503	2521	2531	2539	2543	2549	2551	2557
2579	2591	2593	2609	2617	2621	2633	2647	2657	2659	2663	2671	2677	2683	2687
2689	2693	2699	2707	2711	2713	2719	2729	2731	2741	2749	2753	2767	2777	2789
2791	2797	2801	2803	2819	2833	2837	2843	2851	2857	2861	2879	2887	2897	2903
2909	2917	2927	2939	2953	2957	2963	2969	2971	2999	3001	3011	3019	3023	3037
3041	3049	3061	3067	3079	3083	3089	3109	3119	3121	3137	3163	3167	3169	3181
3187	3191	3203	3209	3217	3221	3229	3251	3253	3257	3259	3271	3299	3301	3307
3313	3319	3323	3329	3331	3343	3347	3359	3361	3371	3373	3389	3391	3407	3413
3433	3449	3457	3461	3463	3467	3469	3491	3499	3511	3517	3527	3529	3533	3539
3541	3547	3557	3559	3571	3581	3583	3593	3607	3613	3617	3623	3631	3637	3643
3659	3671	3673	3677	3691	3697	3701	3709	3719	3727	3733	3739	3761	3767	3769
3779	3793	3797	3803	3821	3823	3833	3847	3851	3853	3863	3877	3881	3889	3907
3911	3917	3919	3923	3929	3931	3943	3947	3967	3989					

(i) $a \cdot b \cdot c$, where a, b, and c are prime and $a \le b \le c$; and
(ii) $a \cdot bc$, where a and bc are prime.

With respect to the latter three-digit condition, numbers such as 211, 311, and 379 are attention-grabbing because they each have at least two children (see **table 3**). This is due either to the fact that we can find a composite child for each of two backward iterations ($211 \leftarrow 2 \cdot 11 = \mathbf{22} \leftarrow 2 \cdot 2 = \mathbf{4}$; $311 \leftarrow 3 \cdot 11 = \mathbf{33} \leftarrow 3 \cdot 3 = \mathbf{9}$) or to the fact that an integer within the backward procedure could have come from different concatenations ($379 \leftarrow 3 \cdot 79 = \mathbf{237} \leftarrow 2 \cdot 3 \cdot 7 = \mathbf{42}$; or $379 \leftarrow 3 \cdot 79 = \mathbf{237} \leftarrow 2 \cdot 37 = \mathbf{74}$). Numbers like 223 are also intriguing because they satisfy either condition, leading back to two children ($223 \leftarrow 2 \cdot 2 \cdot 3 = \mathbf{12}$, or $223 \leftarrow 2 \cdot 23 = \mathbf{46}$). If the backward procedure contains multiple iterations, then multiple children can be found. For example, 773 is the parent to the children 10, 25, 55, and 511:

$$773 \leftarrow 7 \cdot 3 = \mathbf{511} \leftarrow 5 \cdot 11 = \mathbf{55} \leftarrow 5 \cdot 5 = \mathbf{25} \leftarrow 2 \cdot 5 = \mathbf{10}$$

As we extend the number of digits of a prime under consideration, conditions for being a parent become more complicated. We conjecture that four-digit primes of the form $abcd$ can be considered parents if and only if

(i) $a \cdot b \cdot c \cdot d$, where a, b, c, and d are prime and $a \le b \le c \le d$ (e.g., $2237 \leftarrow 2 \cdot 2 \cdot 3 \cdot 7 = \mathbf{84}$);
(ii) $a \cdot b \cdot cd$, where a, b, and cd are prime and $a \le b$ (e.g., $2213 \leftarrow 2 \cdot 2 \cdot 13 = \mathbf{52}$);
(iii) $a \cdot bcd$, where a and bcd are prime (e.g., $3251 \leftarrow 3 \cdot 251 = \mathbf{753} \leftarrow 7 \cdot 53 = \mathbf{371} \leftarrow 3 \cdot 71 = \mathbf{213} \leftarrow 2 \cdot 13 = \mathbf{26}$); or
(iv) $ab \cdot cd$, where ab and cd are prime and $ab \le cd$ (e.g., $3137 \leftarrow 31 \cdot 37 = \mathbf{1147} \leftarrow 11 \cdot 47 = \mathbf{517} \leftarrow 5 \cdot 17 = \mathbf{85}$).

Again, multiple children may be found for a given parent. For example, $3373 \leftarrow 3 \cdot 3 \cdot 73 = \mathbf{657}$ and $3373 \leftarrow 3 \cdot 373 = \mathbf{1119} \leftarrow 11 \cdot 19 = \mathbf{209}$ give us 3373 as the parent of 209, 657, and 1119.

We have not found the sets of conjectures presented here in the literature, but we believe that we have formulated substantial ideas that seem to hold true (albeit without proof). Using our conjectures, we consider the first 550 primes in **table 4**. Gray cells indicate nonparents; blue cells indicate the parents of children between 1 and 100, inclusive (previously found in **table 1**); and white cells indicate parents of at least one composite integer child that is larger than 100 (e.g., 271 is the parent to 142). We have not found anything similar to **table 4** in the literature, and we encourage readers to confirm and build on our work thus far.

POSSIBILITIES FOR RESEARCH

In the twentieth century, computers have provided the tools and calculation techniques for generating data on prime numbers for theorists to consider. Today, computing projects such as the Great Internet Mersenne Prime Search (mersenne.org) continue to produce some of the largest prime numbers. An interesting mathematical fact is that the likelihood that a randomly generated integer having a large number of digits is prime decreases as we consider increasingly larger integers. Yet the number of positive primes is infinite.

Today's mathematicians find advancements in and new findings in home primes exciting and fascinating but also, from a theoretical standpoint, serious for the contribution these findings make to mathematical theory. The home prime conjecture asserts that a prime number (the parent) will be obtained after finitely many PFS iterations on an initial positive integer (the child) other than 1. In other words, every composite number (child) has a home prime (parent) under the PFS process. Heleen (1996–97) and others have actively immersed themselves in attempting to resolve this conjecture for 49, 77, and other composite integers. That is, despite the fact that the PFS process is relatively straightforward, securing home primes for composite integers is an active area of ongoing research in computational number theory.

We encourage readers to have fun building on our discoveries, exploring other conjectures about home primes, or even proving the conjectures that our students made. Possibilities for continued exploration include the consideration of primes larger than four digits, further investigation of home prime families (the existence of multiple children and their identities), and the examination of a modified PFS process that involves concatenating prime factors in reverse order (i.e., in decreasing magnitude, such as $15 = 5 \cdot 3 \rightarrow 53$ [prime]).

BIBLIOGRAPHY

Caldwell, Chris K. 2012. "The Prime Pages: Prime Number Research, Records, and Resources." http://primes.utm.edu/

Caldwell, Chris K., and G. L. Honaker Jr. 2009. "Prime Curios! The Dictionary of Prime Number Trivia." http://www.primecurios.com

DeGeest, Patrick. "Home Primes < 100 and Beyond … Repeated Factorization of Concatenated Prime Factors of Composite Numbers." World of Numbers. http://www.worldofnumbers.com/topic1.htm

Dickinson, Leonard Eugene. 2005. *Divisibility and Primality*, vol. 1. History and Theory of Numbers series. New York: Dover.

Foregger, Thomas, Anthony Morris, and Chi Woo. 2007. "History of Prime Numbers (version 5)." Planetmath.org. http://planetmath.org/HistoryOfPrimeNumbers.html

Great Internet Mersenne Prime Search. http://www.mersenne.org

Heleen, Jeffrey. 1990. "Family Numbers: Mathemagical Black Holes." *Recreational and Educational Computing* 5 (5): 6.

———. 1996–97. "Family Numbers: Constructing Primes by Prime Factor Splicing." *Journal of Recreational Mathematics* 28 (2): 116–19.

Mathematica, Version 8.0. 2010. Champaign-Urbana, IL: Wolfram Research.

MathWorld—a Wolfram Web Resource. http://www.mathworld.wolfram.com

Mersenne Wiki. "Home Prime." http://www.mersennewiki.org/index.php/Home_Prime

———. "Home Primes Search." http://www.mersennewiki.org/index.php/Home_Primes_Search

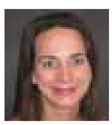
Sloane, Neil, James Alexander, and Jeff Burch. "On-Line Encyclopedia of Integer Sequences: Home Prime Sequence (A037274)." The OEIS Foundation. http://oeis.org/A037274 (see also http://oeis.org/A037274/b037274.txt)

Texas Instruments: Education Technology. TI-89. http://education.ti.com/en/us/home

———. TI-Nspire CAS. http://education.ti.com/en/us/home

Weisstein, Eric W. "Home Prime: MathWorld: A Wolfram Web Resource." http://mathworld.wolfram.com/HomePrime.html

World of Numbers. Recreational Mathematics Forum on Home Primes. http://www.worldofnumbers.com/topic1.htm



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