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They "Get" Fractions as Pies; Now What?

Picture a fifth-grade classroom where students are learning about fractions. Fraction circles are being used to introduce the topic and build on the basic concepts. Near the end of the fraction unit, the teacher asks his students to compare the relative sizes of 3/7 and 7/8. All but three of his students reach for their fraction circles instead of reasoning about the relationships between the two fractions. The teacher had expected many students to compare the given fractions to the benchmark measurement of 1/2, recognizing that 3/7 is less than 1/2 and 7/8 is greater than 1/2.

The teacher had the same uneasy feeling as occurs when students reach for a calculator to complete a simple calculation. He was concerned that his students' use of fraction circles was not helping them internalize fraction concepts in the way that he had intended. This concern resonates with research on learning and teaching fractions.

Researchers caution against using models in a rote way that does not develop students' understanding of the mathematical ideas underlying the models (Clements 2000). Rote use of fraction circles could explain why the students needed the tool to compare 3/7 and 7/8 instead of using understanding derived from the models, as the teacher had intended. Behr, Post, and Lesh (1981) concluded from their studies that students need to interact with multiple models that differ in perceptual features, which causes them to rethink and ultimately generalize the mathematical concepts being investigated with the models.

BUILDING CONCEPTS WITH MULTIPLE MODELS

Evidence abounds that students' learning of skills, concepts, and principles are assisted through the instructional use of manipulatives as well as pictorial models that vary in their features (Fennema 1972; Kieren 1969; and Suydam and Higgins 1977, cited in Behr, Lesh, Post, and Silver 1983). However, the use of models in the classroom should be a means to understanding mathematics—not the end. Research by Fosnot and Dolk (2002) describes how mathematicians use models as mental maps stored in their heads as they solve problems or explore relationships.

Although mathematicians can routinely recall their mental models, students are in the midst of developing both their mental models and their understanding of the concepts. Therefore, many students will need multiple

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Fig. 1 Karen's preassessment and postassessment responses show that the use of a number line helped her solve the problem.

Preassessment

Is $\frac{7}{10}$ closer to . . .

1) 0

I don't know

2) $\frac{1}{2}$

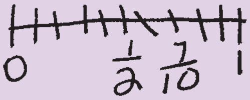
3) 10

4) 17

Postassessment

Is $\frac{7}{10}$ closer to . . .

1) 0



2) $\frac{1}{2}$

3) 10

4) 17

Source: VMAP OGAP 2007

Fig. 2 Mike's solution shows evidence of reasoning that is based on knowledge of the size of the pieces.

Linda hiked 3/7 of the way up Mt. Mansfield. Jen hiked 3/5 of the way up Mt. Mansfield. Who hiked the farthest?

Jenn has hiked more because when the denominator is bigger the pieces are smaller. That means that $\frac{1}{7}$ is smaller than $\frac{1}{5}$ so $\frac{3}{7}$ is smaller than $\frac{3}{5}$ because they have the same numerators.

Source: Petit, Laird, and Marsden 2010

experiences in which they physically construct models or use manipulatives, either commercially prepared or teacher-made, as they solve problems, represent concepts, and ultimately make sense of these ideas.

A recent study by the Vermont Mathematics Partnership Ongoing Assessment Project (VMP OGAP) (2005) revealed the value of emphasizing the use of multiple models, including student-drawn models, to solve problems. The data collected from a sample of fourth-grade students revealed that the percent of students who were able to effectively solve fraction problems using student-generated models increased from 23 percent to 80 percent from preassessment to postassessment.

Strands That Interweave to Form a Foundation

The National Research Council panel identified five interwoven strands that are foundational for students as they develop their understanding of concepts and learn to flexibly and efficiently solve problems (NRC 2001):

1. Conceptual understanding;
2. Procedural fluency;
3. Productive disposition;
4. Adaptive reasoning; and
5. Strategic competence.

Karen's responses to both assessments are shown in **figure 1**. The use of a number line to make sense of the relative magnitude of 7/10 is illustrated in this fourth grader's postassessment response.

Over time, if students are allowed to interact with models whose perceptual attributes vary as well as construct their own models to solve problems, their mental images of, and understandings derived from, the models will be sufficient to solve problems. Mike's solution in **figure 2** illustrates this point. His reasoning about the relative size of "pieces" and his extension of his understanding of unit fractions (1/7 and 1/5) to comparing fractions with the same numerator (3/7 and 3/5) was most likely derived from models.

Figure 3 illustrates a trajectory (i.e., learning sequence) represented in the OGAP Fraction Framework (2008, 2009). When students are guided by

Fig. 3 This trajectory shows the progression of reasoning. The end result, it is hoped, is the use of flexible, efficient, and generalizable strategies.

Karen's Preassessment

Nonfractional reasoning (e.g., used magnitude of the numerator or denominator, not the value of the fraction)

→

Karen's Postassessment

Transitional strategies (e.g., used models)

→

Mike's Response

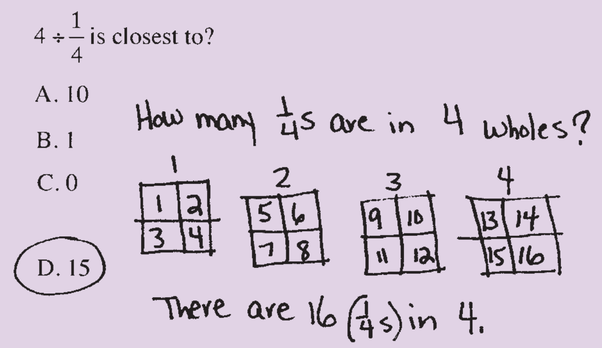
Flexible, efficient, and generalizable fraction strategies (e.g., used unit fraction reasoning)

Source: OGAP Fraction Framework 2008, 2009

knowledgeable teachers, they learn to use models to develop understandings that will help them transition from nonfractional understanding to efficient and generalizable strategies. Notice that Karen's accurate use of a number line (see **fig. 1**) in her postassessment response showed growth from her preassessment. However, Mike's use of unit fractions to compare 3/7 to 3/5 demonstrates a more developed level of understanding. At this level of understanding, students can demonstrate their ability to choose among effective strategies as they compare fractions, such as with the use of benchmarks, unit fractions, equivalence, common denominators, and algorithms. This idea of helping students move along a trajectory that values development of conceptual understanding with a goal of attaining procedural fluency is supported by a National Research Council (2001) panel.

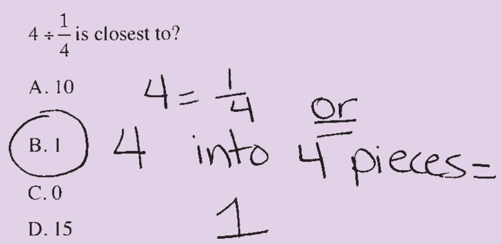
The use of models to develop understanding of fraction concepts is an iterative process. For example, a sixth-grade student, similar to Mike, should no longer need to use a model to compare two given fractions. He should be able to analyze the fractions, the context of the problem, and then flexibly select from a range of strategies to make the comparison. However, as middle school students begin to solve problems involving new concepts, such as multiplication and division of fractions, models can again be used to help bring meaning to these concepts. This idea is demonstrated by Kelyn's work (see **fig. 4a**).

Fig. 4 Students respond differently to a question about division involving fractions. In (a), Kelyn shows a strong understanding. In (b), Cameron relies on procedure rather than meaning of the operation or numbers. In (c), Abdi attempts to calculate an answer after he has developed a model that is sufficient for answering the question.



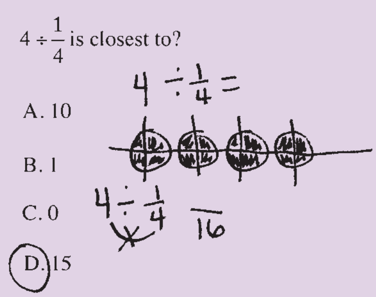
(a) Kelyn

Source: Petit, Laird, and Marsden 2010



(b) Cameron

Source: VMP OGAP 2009



(c) Abdi

Source: VMP OGAP 2009

Using models to highlight the meaning of division should precede the learning of an algorithm for division involving fractions. Researchers agree that students may struggle with the use and understanding of formal algorithms when their knowledge is dependent primarily on memory, rather than anchored within a deep understanding of the foundational concepts (Kieren, as cited in Huinker 2002). For example, Cameron (see **fig. 4b**) seemed to have jumped to a procedure without paying attention to the meaning of the operation or the numbers involved.

If we return to the OGAP Fraction Framework, we find that Cameron is using nonfractional reasoning, whereas Kelyn's (see **fig. 4a**) and Abdi's (see **fig. 4c**) responses show evidence of transitional strategies and an understanding of the meaning of division. Kelyn explicitly restates the problem as "How many 1/4s are in 4 wholes." Abdi creates an effective area model that shows the answer 16, then goes beyond his model to attempt a calculation.

Kelyn's and Abdi's use of models to solve these division problems provides a teacher with the opportunity to capitalize on understandings evidenced in their work. Progressing toward more generalized and efficient approaches or to a deeper conceptualization of a mathematical idea does not come simply from the use of models alone, rather from "The teacher makes the system work" (Hiebert et al. 1997, p. 41). The teacher is responsible for providing tasks that engage students in the mathematical concepts and then engineering discussions to bring out the mathematics represented in the models (Hiebert et al. 1997).

Fig. 5 Alex uses the "common denominator" algorithm for solving this division problem.

Jim is making decorations for a school dance. He has 4 1/4 yards of wire. Each decoration needs 3/4 of a yard of wire.

1) How many full decorations can Jim make from 4 1/4 yards of wire? Show your work.

$\frac{17}{4} \div \frac{3}{4} = \frac{5\frac{2}{3}}{1} = 5\frac{2}{3}$ he can make 5 full decorations

Source: VMP OGAP 2009

Fig. 6 Kim's invert-and-multiply strategy helps her solve the problem.

Mrs. Jenkins has 2 3/4 gallons of juice for the class picnic. The juice is divided equally into four pitchers. How much juice is in each pitcher? Show your work.

$2\frac{3}{4} \div 4$

$\frac{11}{4} \times \frac{1}{4} = \frac{11}{16}$ of a gallon in each pitcher.

Source: VMP OGAP 2009

TIPS FOR CLASSROOM PRACTICE

The real challenge is to use models to help students develop a deep understanding of concepts and relationships and to move the students to the use of efficient and generalizable strategies for solving problems. For example, a teacher might want to build on Kelyn's and Abdi's understandings about division of a whole number by a unit fraction. To do so, the teacher may lead off a discussion using their solutions as a way to engage *all* students in developing further their understanding of the meaning and impact of dividing by a fraction. In particular, he or she could show these two solutions and pose questions that help all students analyze aspects of the models that reinforce a deeper, more complete meaning of division. A set of well-engineered questions could be posed to extend their reasoning beyond unit fractions, such as the following:

- Your model shows that there are 4 one-fourths in every whole. How many one-fourths do you think are in 5 (or 6, or 10, or 100)?
- How many thirds, fifths, or sixths are in 4 (or 5, or 6, or 100)?
- What patterns do you see?
- Make and test a conjecture about the patterns that you see. (Give students the chance to say, "I noticed that . . .") (Petit, Laird, and Marsden 2010).

As students begin to see patterns and relationships when dividing by a unit fraction, it is important to pose questions that extend the investigation beyond unit fractions and include dividends that are fractions and divisors that are not unit fractions, such as these:

- How many three-fourths are in 3 (or 6; or 3/4; or 1 1/2)?
- Make and test a conjecture about the patterns that you see.

Fig. 7 This trajectory runs the gamut from nonfractional reasoning to flexible, efficient, and generalizable strategies as applied to the division of fractions.

Cameron's Response		Kelyn's and Abdi's Responses		Kim's and Alex's Responses
Nonfractional reasoning (e.g., applied a procedure without attention to the meaning of division)	⇒	**Transitional strategies** (e.g., used models and showed some evidence of understanding the impact of division by fraction)	⇒	**Flexible, efficient, and generalizable fraction strategies** (e.g., used common-denominator or invert-and-multiply algorithm)

Source: OGAP Fraction Framework 2008, 2009

Because mathematics can be considered a study of patterns, students' success in mathematics will be based, in part, on their ability to organize information in a way that makes patterns obvious. For example, students investigating the questions based on Kelyn's and Abdi's models may be more likely to see patterns and relationships *as they develop* if they organize their information in a table.

Over time, as students make observations, test conjectures, and discover patterns and relationships, they should develop efficient and generalizable strategies that they can apply to problem situations, such as what occurs with Alex's and Kim's work in **figures 5** and **6,** respectively. Alex uses the common-denominator algorithm, and Kim employs the invert-and-multiply algorithm.

Ultimately, one wants to know whether students can apply efficient fractional strategies across a range of problem situations. However, we use these single samples of student work in **figure 7** to further describe the OGAP Fraction Framework first introduced in **figure 3** to highlight a trajectory from nonfractional reasoning to flexible and efficient strategies, in this case, for the division of fractions.

Teachers involved in OGAP reported that using the trajectory described in the OGAP Fraction Framework, and the research that underpins the Framework, has—

1. improved their capacity to recognize students' mathematical understanding and misunderstanding in student written work and in-class discussions; and
2. empowered them to make more informed instructional decisions.

The real challenge is to use models to help students develop a deep understanding of concepts and relationships and move them to use efficient strategies.

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Marjorie M. Petit, mpetit@gmavt.net is an independent consultant in mathematics curriculum, instruction, and assessment. **Robert Laird,** rlaird@uvm.edu, is a mathematics professor at the University of Vermont. **Edwin Marsden** was also a mathematics professor at the University of Vermont. All three authors played major roles in the development of the Vermont Mathematics Partnership Ongoing Assessment Project (OGAP). Petit and Laird would like to dedicate this article to their coauthor, **Edwin Marsden**, who passed away in June 2010. Marsden worked in the K–12 system providing professional development and in schools and classrooms with teachers and students. It was not uncommon to see this man, who had a huge impact on Vermont mathematics, with a bunch of elementary students sitting around a table solving a problem.

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