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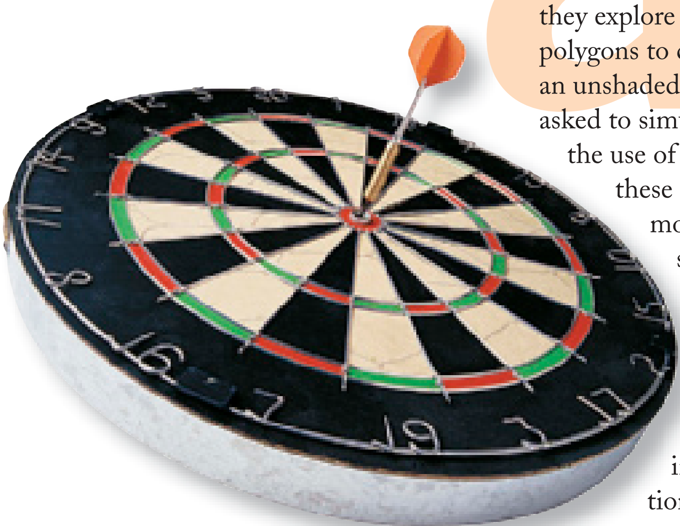
Target Geometry and Probability Using a Dartboard



Edited by **Gwen Johnson**, gwendolyn.johnson@unt.edu, University of North Texas, Dallas, and **James Dogbey**, jdogbey@clemson.edu, Clemson University, South Carolina. This department's classroom-ready activities may be reproduced. Teachers are encouraged to submit manuscripts in a format similar to this department based on successful activities from their own classes. Those activities focusing on NCTM's Content and Process Standards and Focal Points and problems with a historical foundation are of interest. Send submissions by accessing **mtms.msubmit.net**.

As middle-grades students explore area problems in geometry class, they typically calculate areas of circles and squares using formulas. Sometimes they explore problems using special polygons to compare a shaded area to an unshaded area. Students are often asked to simulate probability through the use of coins and dice. However, these situations contain little motivation for finding a solution. The following dartboard activity investigates how students can use simulations to determine experimental probability and build on intuition to make connections between calculating area and finding the theoretical probability of random events.

We present three different dartboards (A, B, and C) that promote this building of connections between geometric and probabilistic thinking in middle-grades students. We created our dartboards using The Geometer's Sketchpad®, version 4.0.

For each dartboard style, students will—

1. simulate throwing darts and determine the experimental probability of hitting the gray and black areas;
2. investigate and estimate probability through the counting of squares; and
3. use area formulas to calculate the theoretical probability.

This activity provides a link between geometry and probability, so that students can understand how mathematical topics are connected across the curriculum. It incorporates mathematical modeling, which encourages students to focus on relationships between simulations and mathematical formulas as advocated in *Principles and Standards for School Mathematics* (NCTM 2000).

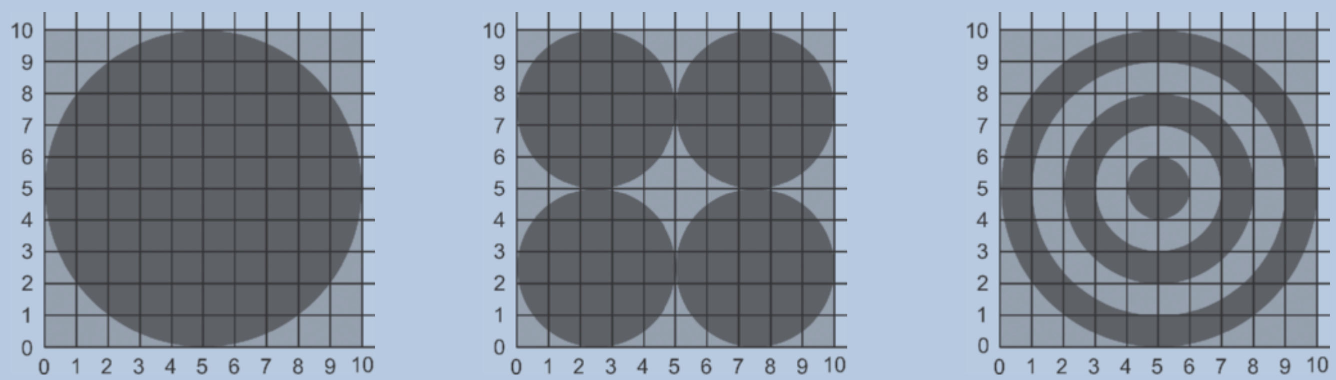
INTRODUCING THE ACTIVITY

Before introducing the dartboards, students should analyze the random-digit table (see **table 1**). The table of numbers simulates rolling a die and generating random numbers. The table's rows and columns of random digits are grouped in fives. It can be read either horizontally or vertically. Many of the students in the eighth-grade class had not encountered such a random-digit table, so we asked them what they thought it was. Students immediately began making observations about the table and we heard these remarks:

Table 1 A random-digit table simulates randomly selecting digits from 0 to 9. In this table, the digits are grouped into five-digit clusters.

Row	Column							
	1	2	3	4	5	6	7	8
1	10480	15011	01536	02011	81647	91646	69179	14194
2	22368	46573	25595	85393	30995	89198	27982	53402
3	24130	48360	22527	97265	76393	64809	15179	24830
4	42167	93093	06243	61680	07856	16376	39440	53537
5	37570	39975	81837	16656	06121	91782	60468	81305
6	77921	06907	11008	42751	27756	53498	18602	70659
7	99562	72905	56420	69994	98872	31016	71194	18738
8	96301	91977	05463	07972	18876	20922	94595	56869
9	89579	14342	63661	10281	17453	18103	57740	84378
10	85475	36857	53342	53988	53060	59533	38867	62300

Fig. 1 Three different dartboard styles resulted in different probability results.



- It is a table filled with numbers.
- They are all random.
- Five numbers are grouped together.

The purpose of the random-digit table was to generate the experimental probability of a randomly thrown dart hitting either gray or black on the dartboard. Four digits were selected at random, which represented the coordinates of the flight of the dart. For example, if row 4, column 2 was chosen (see **table 1**), the digits 9309 would be recorded. As a class, we discussed that we only needed four of the five numbers in a given cell to determine the location of the dart.

To help students use the table, we posed this question: "Suppose I selected these four digits, 9309, out of the random-digit table. What are the coordinates determined from these digits for the dart?" Students analyzed dartboard A (see **fig. 1a**) and contemplated how they could determine coordinates for the dart. Most students immediately recognized that the numbers could be split into an ordered pair (93, 09).

When we asked them to further scrutinize the dartboard to determine the intervals for the coordinates, they realized that since the x-axis and y-axis ranged from 0 to 10, they needed to place a decimal point between the digits (9.3, 0.9) . We then asked them, "What color did the dart hit?" They answered, "Gray!" For each style of dartboard, students needed to simulate "throwing" 10 darts by extracting ten groups of four digits from the table. They recorded the random digits and then determined the coordinates for the dart and the color the dart hit.

DARTBOARD A

Before completing the simulations, we asked students to predict the probability of hitting gray and black and compare their predictions with the theoretical probability calculated using the geometric models. For dartboard A, the most common prediction was

"80 percent of the time, black was hit; and 20 percent of the time, gray was hit." At the beginning of the activity, students should work with a partner to determine the random digits, the dart coordinates, and the color hit. One student was the recorder, and the other student "threw" the dart, which meant that they had to use the coordinates to locate where the dart hit and identify the color. Several students marked the dartboard with points to show where the dart hit.

After completing the simulations, students should have identified the experimental probability of hitting the gray and black areas. Students shared their solutions, which led to the realization that the two events, hitting gray versus hitting black, were complementary. *Complementary events* are two actions that make up the entire sample space and are mutually exclusive, which means they have no outcomes in common.

When we discussed how the probability for each color could also be determined by counting squares, some students realized that they would only need to count one color and subtract this number from 100 (since there are 100 squares total). Several students also pointed out that dartboard A had symmetry, that only one corner of gray squares needed to be counted, and that the result could be multiplied by 4.

In determining the theoretical probability, students had no difficulties calculating the area of the circle and the square. They realized that finding the difference of these two areas would determine the theoretical probability of a randomly thrown dart hitting the gray area.

Generate Random Numbers on a TI Calculator

Instead of using the random-digit table to generate ordered pairs, a TI-84 graphing calculator's random-number generator can be used. These are the keystrokes:

1. MATH→ PRB→randInt(
2. Type in randInt (0, 100, 2) to give random ordered pairs between 0 and 100.
3. Each time you press the ENTER key, an new pair of random numbers is generated.
4. To use these integer pairs to get numbers in the desired range, place a decimal point between the first and second digit. For example {23, 4} would be point (2.3, 0.4).

DARTBOARD B

Students were eager to begin simulating throwing darts at dartboard B (see **fig. 1b**). Before they began, they needed to make predictions. Most students predicted that gray would be hit 35 percent of the time and black, 65 percent, commenting, "I think this board will have more gray than the other [dartboard A]." A few students predicted that there was more black area on dartboard B: "There's no box that is completely gray." Students were using new strategies that incorporated their geometrical reasoning to help them make probability predictions. However, the theoretical probabilities for A and B were exactly the same. To calculate the theoretical probabilities, students determined that the diameter of each small circle was five units and then calculated the area for each of the small black circles.

After students finished calculating the theoretical probability for dartboard B, they needed to give advice about using dartboard A versus B. Most students realized that the theoretical results were the same but commented about the different aesthetic styles of the boards:

> Although the dartboards are the same probabilities, dartboard B is more spaced out and may be harder to hit [black]. I would choose A. It is easier to hit a bigger target.
>
> I think you should use B because it looks like there is more gray than there really is.
>
> You can pick either one because they are the same. B has more circles to choose from, but A has a bigger target.
>
> I liked the second one because it is a more unique board.

DARTBOARD C

Dartboard C, the last discussed, was made of concentric circles (see **fig. 1c**). When students estimated the probability for hitting each color with a randomly thrown dart, they commented on the uniqueness of the design. Typically, students predicted that both colors had equal probability. Since this was the third dartboard, students were quick to finish the simulations and count the squares. When it came to calculating the theoretical probability, students found the task challenging. They had difficulty calculating the first gray ring. It was hard for them to keep all the calculations organized, so we discussed how we could number the concentric circles starting with the inner-black circle (the bull's-eye), which we labeled as black #1. **Table 2** helped us organize our remaining calculations.

At this point, students were able to calculate the remaining areas and determine the theoretical probability for landing on each color. As they completed this activity for dartboard C,

Table 2 A student found these results when organizing information for concentric circles.

Area to be Calculated	Radius	Area of Circle (Units^2)	Area of Inner Part (Subtract) (Units^2)	Final Calculated Area (Units^2)
$A_{gray\#1}$	2	$3.14(2)^2 = 12.56$	3.14	9.42
$A_{black\#2}$	3	$3.14(3)^2 = 28.26$	12.46	15.7
$A_{gray\#2}$	4	$3.14(4)^2 = 50.24$	28.26	21.98
$A_{black\#3}$	5	$3.14(5)^2 = 78.5$	50.24	28.26

they concluded that the theoretical probability for each color was similar (53 percent gray and 47 percent black).

When students were asked, "Which method of estimating the probability, simulating the darts or counting the squares, produced better results?" their thinking was divided. Some students believed that using simulations was better: "Simulating the darts produced better results because it was random and throwing darts is random." Other students reiterated this same idea by explaining that simulations were more like "actually playing on the board." Another student preferred counting the squares: "It is easier and you can go back and double check." This student compared the two results with the theoretical probability and used this reasoning to explain which method was better:

> Counting the squares was the most accurate for me because when we were doing the simulating darts, we had some off results, like so-so, which were way off the actual ones [theoretical]. We were also a lot closer when counting the squares versus simulating.

The last question in the activity asked, "If a dart club chose C for its spring tournament, what mathematical thinking did they use?" Most students commented that the dart club wanted to have "a more even chance of hitting black or gray," meaning that the probabilities of each outcome were close to 50 percent. Using similar reasoning, some students believed that dartboard C was "fair." One student commented that the dart club used probability to make its decision:

> Based on my results, there's a 50 percent chance you'll hit grey, and a 50 percent chance you'll hit black, so it's not based on ability, it's based on probability.

CONCLUSION

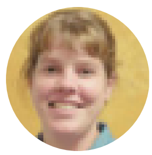
As students simulate throwing darts and determining the experimental probability for each dartboard, pose these questions:

- Does it matter how random digits are selected from the table?
- When a person throws a dart at a board, is it truly random?

Middle-grades students need to encounter situations where connections can be made between content areas so that they can build on their mathematical reasoning. *Principles and Standards* advocates that students be exposed to rich problems so that students can build mathematical models that relate to real life. Most eighth-grade students understood how to play darts and realized that the simulation was similar to actually throwing a dart at a dartboard. In addition, they understood that calculating the areas allowed them to find the theoretical probability, which is important in understanding the "fairness" of the game. Throughout this activity, students connected geometry with probability to analyze the different dartboard styles.

REFERENCE

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, VA: NCTM, 2000.



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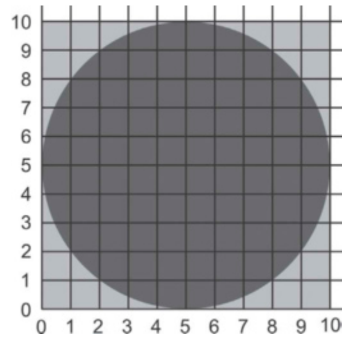
Solutions to **activity sheets 1, 2,** and **3** are appended online to "Mathematical Explorations" at **www.nctm.org/mtms.**

activity sheet 1

Name ______________________________

DARTBOARD A



1. Predict the probability of a randomly thrown dart hitting these areas:

P(*gray*) = ____________________

P(*black*) = ____________________

2. Work with a partner and complete 10 simulations for randomly throwing darts at the dartboard. Select your digits, and determine the coordinates for each dart and the color hit.

Simulations

Random Digits	Coordinates (*x*, *y*)	Color That Dart Hit

3. Based on your results, fill in these blanks. *P*(*gray*) = ________ *P*(*black*) = ________

4. We can also estimate the probability by counting the number of squares of each color divided by the total number of squares. Based on this estimate, fill in these blanks. *P*(*gray*) = ________ *P*(*black*) = ________

5. Since the dartboard consists of geometric shapes, we can also calculate the true theoretical probability by determining the area of the square and the circle. Reminder: The area of a circle is $A = \pi(r)^2$, where r is the radius of the circle.

a. What is the area of the square? ________________

b. What is the radius of the black circle? ________________

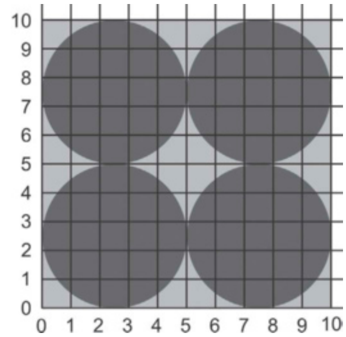
c. If we use 3.14 for π, find the following areas: A_{black} = ____________ A_{gray} = ____________

Based on the areas, what is the theoretical probability: *P*(*gray*) = _____ *P*(*black*) = _____

activity sheet 2

Name ______________________________

DARTBOARD B



1. Predict the probability of a randomly thrown dart hitting these two areas:

P(*gray*) = ____________________

P(*black*) = ____________________

2. Work with a partner and complete 10 simulations for randomly throwing darts at the dartboard. Select your digits, and determine the coordinates for each dart and the color hit.

Simulations

Random Digits	Coordinates (x, y)	Color That Dart Hit

3. Based on your results, fill in these blanks. *P*(*gray*) = ________ *P*(*black*) = ________

4. We can also estimate the probability by counting the number of squares of each color divided by the total number of squares. Based on this estimate, fill in these blanks. *P*(*gray*) = ________ *P*(*black*) = ________

5. Since the dartboard consists of geometric shapes, we can also calculate the true theoretical probability.

a. What is the radius of each small black circle? ______________

b. How many small black circles are there? __________

c. Using this information, we can determine the areas of the following:

A_{black} = ____________ A_{gray} = ____________

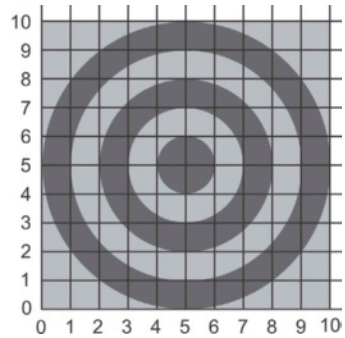
Based on the areas, what is the theoretical probability: *P*(*gray*) = _____ *P*(*black*) = _____

6. Using your experimental and theoretical results, what advice would you give about using dartboard A versus B? Explain your thinking.

activity sheet 3

Name ______________________________

DARTBOARD C



1. Predict the probability of a randomly thrown dart hitting the following:

$P(gray)$ = ____________________

$P(black)$ = ____________________

2. Work with a partner and complete 10 simulations for randomly throwing darts at the dartboard. Select your digits, and determine the coordinates for each dart and and the color hit. On a separate sheet of paper, create a Simulations table as shown on **activity sheet 1** and **activity sheet 2**.

3. Based on your results, fill in these blanks. $P(gray)$ = _________ $P(black)$ = _________

4. We can also estimate the probability by counting the number of squares of each color divided by the total number of squares. Based on this estimate, fill in these blanks. $P(gray)$ = _________ $P(black)$ = _________

5. If we number the concentric circles, starting with the black bull's-eye, we can calculate the true theoretical probability by determining these two numbers: A_{square} = _________ $A_{black\#1}$ = _________

6. For these area calculations, determine the radius and subtract the inside area.

7. For the last area calculation, $A_{gray\ \#3}$, we need to determine the following answers:

a. What is the area of the square? ____________

b. What is the area of the inner part that needs to be subtracted? _________

c. The final calculated area is

____

Area to be Calculated	Radius	Area of Circle (Units^2)	Area of Inner Part (Subtract Units^2)	Final Calculated Area (Units^2)
$A_{gray\#1}$				
$A_{black\#2}$				
$A_{gray\#2}$				
$A_{black\#3}$				

Based on the areas, what is the theoretical probability: $P(gray)$ = _____ $P(black)$ = _____

8. Which method of estimating the probability, simulating the darts or counting the squares, produced better results? Explain your thinking.

9. If a dart club chose dartboard C for its spring tournament, what mathematical thinking did they use? Explain.

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SOLUTIONS TO THE ACTIVITY SHEETS

Dartboard A

1. Students' predictions will vary.

2. The simulations will yield various results.

3. We found that most of the students' results were around $P(gray)$ = 2/10, or 20%; and $P(black)$ = 8/10, or 80%.

4. The number of gray squares is approximately 22 out of 100, or $P(gray)$ = 22/100, or 22%; and the number of black squares is approximately 78 out of 100, or $P(black)$ = 78/100, or 78%.

5. The theoretical probability can be found by:

a. $A_{square} = 10 \times 10 = 100$ units^2
b. The radius is 5 units.
c. $A_{black} = \pi(5)^2 = 25\pi \approx 78.5$ units^2, and $A_{gray} = 100 - 25\pi \approx 21.5$ units^2

$$P(gray) \approx \frac{21.5}{100} = 21.5\%$$
$$P(black) \approx \frac{78.5}{100} = 78.5\%$$

Dartboard B

1. Students' predictions will vary.

2. The simulations will yield various results.

3. We found that most of the students' results were around $P(gray)$ = 2/10, or 20%; and $P(black)$ = 8/10, or 80%.

4. The number of gray squares is approximately 22 out of 100, or $P(gray)$ = 22/100, or 22%; and the number of black squares is approximately 78 out of 100, or $P(black)$ = 78/100, or 78%.

5. The true theoretical probability can be found by the following:

a. The radius of each circle is 2.5 units.
b. There are four small black circles.
c. $A_{black} = 4\pi(2.5)^2 = 25\pi \approx 78.5$ units^2, and $A_{gray} = 100 - 25\pi \approx 21.5$ units^2

$$P(gray) \approx \frac{21.5}{100} = 21.5\%$$
$$P(black) \approx \frac{78.5}{100} = 78.5\%$$

6. For both A and B, the theoretical results are identical. Also, the experimental results are similar. Based on these data, the dart club could choose either A or B and have similar results.

Dartboard C

1. Students' predictions will vary.

2. The simulations will yield various results.

3. We found that most of the students' results were around $P(gray)$ = 5/10, or 50%; and $P(black)$ = 5/10, or 50%.

4. The number of gray squares is approximately 49 out of 100, or $P(gray)$ = 49/100, or 49%; and the number of black squares is approximately 51 out of 100, or $P(black)$ = 51/100, or 51%.

5. The theoretical probability can be found by the following:

$A_{square} = 10 \times 10 = 100$ units^2
$A_{black\#1} = \pi(1)^2 = \pi \approx 3.14$ units^2 (since the radius is 1)

6. The table should appear as follows:

Area to be Calculated	Radius	Area of Circle (Units²)	Area of Inner Part (Subtract Units²)	Final Calculated Area (Units²)
$A_{gray\#1}$	2	4π	π	3π
$A_{black\#2}$	3	9π	4π	5π
$A_{gray\#2}$	4	16π	9π	7π
$A_{black\#3}$	5	25π	16π	9π

7. The last area calculations are as follows:

a. $A_{square} = 10 \times 10 = 100$ units^2
b. The area of the inner part is 25π units^2
c. The final calculated area is $A_{gray\#3} = 100 - 25\pi \approx 21.5$ units^2.

Based on the areas, the theoretical probability is this:

$$\begin{aligned} P(gray) &= \frac{3\pi + 7\pi + 100 - 25\pi}{100} \\ &\approx \frac{52.88}{100} = 52.9\% \\ P(black) &= \frac{\pi + 5\pi + 9\pi}{100} \\ &\approx \frac{47.1}{100} \\ &= 47.1\% \end{aligned}$$

8. Students' answers will vary based on preference and accuracy of counting boxes.

9. Based on these data, I assume that the dart club was looking for a board that had similar probabilities for hitting black and gray. (Or maybe club members really liked the concentric-circle look of a dartboard!)