What a Pip! Probability and Efron's Dice

Ben C. Sloop and S. Megan Che

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Edited by **Gwen Johnson**, gwendolyn.johnson@unt.edu, University of North Texas, Dallas, and **James Dogbey**, jdogbey@clemson.edu, Clemson University, South Carolina. Readers are encouraged to submit manuscripts through **http://mtms.msubmit.net**.

Students in grades 6–8 need to develop an understanding of notions of chance. Teachers can promote this chance understanding by assigning probability experiments in which students make predictions and test conjectures (NCTM 2000). These experiments usually involve rolling a die, flipping a coin, or using a spinner in which each outcome is equally likely. The activity discussed here, however, builds on students' sense of fairness as they explore chance using a set of nontraditional dice with some interesting properties.

Efron's dice (Gardner 1970), a set of four nontraditional dice, were created by Bradley Efron, a statistician at Stanford University. See the diagram in **figure 1**: Four faces of die A have 4 *pips*, or dots, and two faces are blank. All six faces of die B contain 3 pips each. Four faces of die C have 2 pips each, and two faces have 6 pips each. Three faces of die D have 1 pip each, and three faces have 5 pips each.

Consider, for instance, a game in which students roll die A and die B; the die producing the larger number is the winner. Die A will win when any of the four faces with 4 pips is rolled, because die B always produces a 3. Die A loses when either of its two blank faces is rolled. On average, we would expect die A to win in 4 out of 6 rolls. Thus, die A is more likely to roll a number larger than die B. The probability that die A wins over die B is two-thirds, denoted as

$$P(\mathrm{A} > \mathrm{B}) = \frac{2}{3}.$$

Similarly, die B has an advantage over die C, and die C has an advantage over die D. However, these dice are nontransitive in that die D has an advantage over die A. As students quantify the advantages in certain pairs of dice, they see that these dice were constructed such that

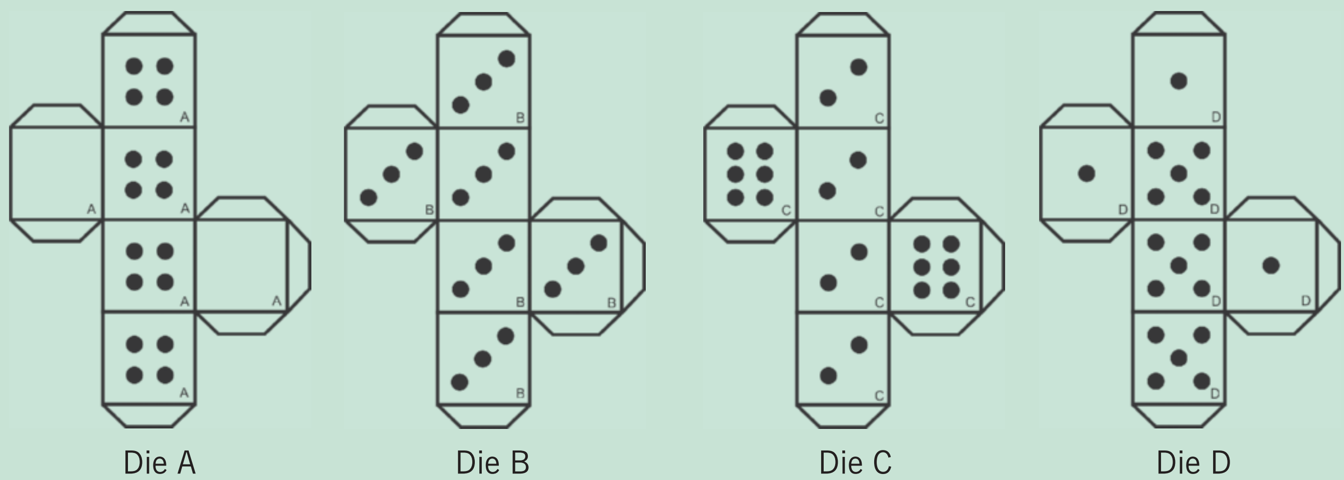
$$P(\mathrm{A} > \mathrm{B}) = P(\mathrm{B} > \mathrm{C}) = P(\mathrm{C} > \mathrm{D}) = P(\mathrm{D} > \mathrm{A}) = \frac{2}{3}.$$

LAUNCHING THE DICE

The questions and discussions that follow from **activity sheet 1** allow the teacher to assess and draw on students' prior knowledge regarding—

- experimental versus theoretical probabilities;
- sample space;
- how to represent single-stage probabilities as fractions; and
- probability notation.

Fig. 1 Efron's dice provide interesting explorations of probability.



With this activity, students work in pairs; one student is given die A and the other die B. Students are told that the player who rolls the higher number is the winner and are asked to decide and justify whether this is a fair game. Further, if students determine that one player has an advantage, the teacher can challenge students to quantify this advantage.

To understand students' thinking and reasoning about the task, a teacher can facilitate a guided discussion using the following questions:

- I noticed that many students stopped rolling die B and only rolled die A. Why?
- How did you determine which die had the advantage? How can you be certain?
- Suppose you play the game 10 times. Is it possible that die B will win more often? Is it probable? What if you played 100 times? Played 1000 times?
- What are all the possible outcomes of this game?

When the class addresses the possible outcomes, it may be helpful for the teacher, or the pair of students sharing their strategy, to list the entire sample space and discuss whether all thirty-six outcomes are possible—twenty-four occurrences in which A = 4 and B = 3, and twelve occurrences where A = 0, B = 3—or two outcomes: A = 4 or A = 0. In our experience, students see the possibilities both ways; it is helpful if students can use a tabular format to represent all thirty-six possibilities.

Before moving to compare die B and die C in question 11 of **activity sheet 1**, students are asked to make a prediction about which die would have an advantage. Students will often predict that die C, with a total of 20 pips, will beat die B more often, because this die has a sum of only 18 pips. Students' intuition regarding spreading the total number of pips over the six possible sides will be helpful in extension 2 (see the **sidebar** on p. 118) in which students are introduced to expected value. However, for this game, the players are only interested in whether or not the number is greater; the magnitude of the difference in the two dice is not important.

Next, student pairs are given die B and die C and asked to test their conjectures. Students usually jump directly to the theoretical probability because they have already seen that rolling die B does not affect the result of the game. Groups should compare results and strategies and justify their findings.

After groups share their justifications, the following questions may be posed to the class:

- Some groups found *P*(B > C) to be 2/3, whereas others determined it to be 4/6. Which finding is correct?
- Suppose I never gave you die C but told you the probability that die B wins is 2/3. Could you determine the probability that die C wins? How?

Introduce to students the concepts of *complementary* and *mutually exclusive* events if these topics have not been discussed. Pose the following scenarios for student experiments:

1. Die C is rolled once. Die C rolling a 2 is one event, and die C rolling a 6 is another event.
2. A traditional six-sided die with the numbers 1 through 6 is rolled once. A result of 2 is one event, and a result of 6 is another event.
3. Player 1 rolls die B, and player 2 rolls die C. Die B rolling a 3 is one event, and die C rolling a 6 is another event.
4. Player 1 rolls die A, and player 2

Extensions

1. In pairs, students could play a game in which one student selects one of the four dice to play and the other student randomly draws one of the remaining dice. Both dice are rolled, and the die that produces the higher number wins the game. If the task is unclear, the teacher may wish to simulate a few rounds in which a student selects a die and he or she blindly chooses and rolls one of the remaining dice. Groups are then challenged to determine which die would be the best choice. Given the computationally intensive nature of this extension, the teacher may wish to use larger groups or begin the extension as a class and delegate the exploration of particular dice to various groups. Groups will be instructed to organize their findings using mathematical evidence to support their choice. As it turns out, die C has a slight advantage,14/27, over the other three dice.

2. The teacher may choose to alter the game so that players take the sum of multiple rolls before determining the winner. In this version, the magnitude of the difference in a single roll is important. As students are challenged to determine the best die, they are introduced to ideas of expected value. Students usually claim that die C is the best because the total number of points on this die is the largest.

With a bit of guidance from the teacher, students begin to develop the concept of expected value. In this situation, it makes sense to spread the total number of pips evenly over the six sides to determine the average roll for a die. Students do not generally struggle to see that, on average, die B receives 3 points because all the sides have 3 pips. However, students are sometimes uncomfortable with the fact that die D also has an expected return of 3 points, despite the fact that none of its sides have 3 pips.

3. Students may wish to explore two other sets of nontransitive dice, also created by Bradley Efron (Gardner 1970). These dice have fewer repeated numbers, which make the analysis of the dice more challenging.

Set 1

A = {2, 3, 3, 9, 10 11}
B = {0, 1, 7, 8, 8, 8}
C = {5, 5, 6, 6, 6, 6}
D = {4, 4, 4, 4, 12, 12}

Set 2

A = {1, 2, 3, 9, 10, 11}
B = {0, 1, 7, 8, 8, 9}
C = {5, 5, 6, 6, 7, 7}
D = {3, 4, 4, 5, 11, 12}

Because a tie is possible when rolling the second set of dice, students can explore how another roll after a tie will affect the resulting probabilities.

rolls die C. Die A rolling a 4 is one event, and die C rolling a 6 is another event.

After the experiments 1–4 above, ask students if the two events are complementary and mutually exclusive, and why. Then present this scenario:

5. In previous discussions, we said, “If two events are complementary, then the sum of their probabilities is 1.” In experiment 4, $P(\mathrm{A} = 4) = 2/3$, and $P(\mathrm{C} = 6) = 1/3$. Here we have

$$P(\mathrm{A}=4)+P(\mathrm{C}=6)=\frac{2}{3}+\frac{1}{3}=1.$$

Why, then, aren’t these two events complementary?

In experiment 1 in which die C is rolled, a 6 could not be rolled if a 2 is rolled. These events are (1) mutually exclusive, because they cannot happen simultaneously (the only options for die C are to roll a 2 or roll a 6); (2) exhaustive, because they cover all possible outcomes of the experiment; and (3) complementary, because not getting a 2 means that a 6 was rolled.

In experiment 2, in which a traditional die is rolled once, rolling a 2 and rolling a 6 are mutually exclusive events because they cannot happen simultaneously. They are not exhaustive, because it is possible that a different number, specifically 1, 3, 4, or 5, can be rolled. These two events are not complementary, because not rolling a 2 does not necessarily mean a 6 was rolled. Students see that although mutual exclusivity is a necessary criterion for events to be complimentary, it is not sufficient; the two events must also cover all possibilities for the experiment.

In experiment 3, in which die B and die C are rolled, these events are not mutually exclusive (because player 1 could roll a 3 at the same time that player 2 rolls a 6) and are not complementary. Students also notice that

$$P(\mathrm{B}=3)+P(\mathrm{C}=6)=1+\frac{1}{3}\neq 1,$$

which implies that the two events are not complementary.

The interesting point with experiment 4, in which die A and die C are rolled, is that the probabilities of these two events sum to 1, but these events are not complementary. Thus, this question can perturb students who may assume that whenever the probabilities of two events sum to 1 those events are complementary.

EXPLORING TWO-STAGE PROBABILITIES

In **activity sheet 2**, students will be challenged to determine if die C or die D has an advantage and asked to find the theoretical probability that die C rolls a number greater than die D. Pairs will be asked to justify their conclusions and should organize their evidence to share with the class. This task stretches students’ thinking because the outcome of this game is dependent on the results of both dice. In our experience with this task, we have seen variations of the following four solution strategies.

Solution 1

Some students write out the entire sample space to determine that die C wins in 24 of the 36 outcomes, which is similar to **figure 2**. Some students simply see the total number of wins, whereas others may examine the three cases in which die C beats die D separately: (C = 6, D = 5), (C = 6, D = 1), and (C = 2, D = 1). Therefore,

$$P(\mathrm{C} > \mathrm{D}) = \frac{24}{36} = \frac{2}{3}.$$

Solution 2

After a few trials of the game, some students realize that the only way die D can win is when C = 2 and D = 5. These students simplify the number of cases to consider by first finding $P(\mathrm{C} > \mathrm{D})$ and using its complement to determine that $P(\mathrm{C} < \mathrm{D}) = 1 - P(\mathrm{C} > \mathrm{D})$.

Solution 3

Similar to solution 2, some students determine $P(\mathrm{C} > \mathrm{D})$ by multiplying

$$P(\mathrm{C} = 2) \times P(\mathrm{D} = 5) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}.$$

It is important that students justify their decision to multiply these probabilities. Students' justifications often reveal that, on average, die D rolls a 5 one-half the time. Within this half, C rolls a 2 two-thirds of the time, on average. Therefore, C = 2 when D = 5 two-thirds of one-half the time, or

$$\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

of the time. The teacher may wish to connect the sample-space table in **figure 2** to an area model of multiplication, as in **figure 3**, to further illustrate this strategy.

Fig. 2 The sample space for die C versus die D is illustrated.

	6	6	2	2	2	2
5	✓	✓	✗	✗	✗	✗
5	✓	✓	✗	✗	✗	✗
5	✓	✓	✗	✗	✗	✗
1	✓	✓	✓	✓	✓	✓
1	✓	✓	✓	✓	✓	✓
1	✓	✓	✓	✓	✓	✓

Fig. 3 An area model diagrams the intersection denoted by $P(C = 2 \cap D = 5)$.

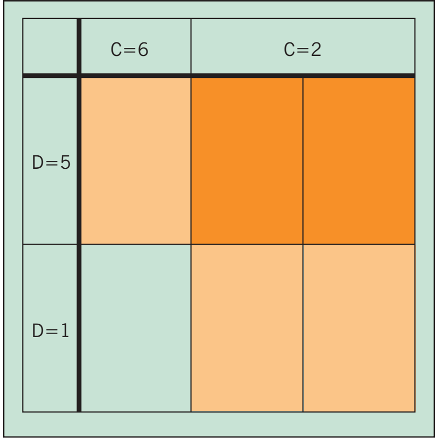
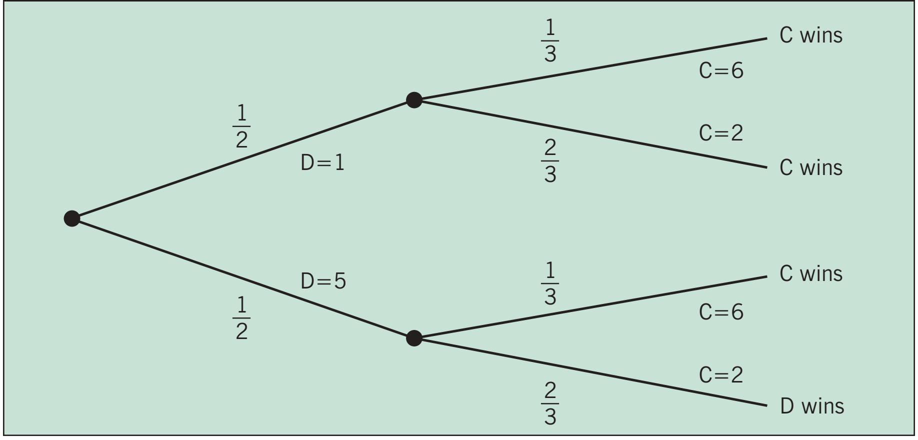


Fig. 4 This tree diagram, outlining the winners when die C is played against die D, resembles a March Madness basketball bracket.



Solution 4

Other students discover ideas of conditional probability when they see that die C's probability of winning is dependent on the outcome of die D. This often arises in students' discussions that are similar to the following: If die D rolls a 1, then die C wins, but if die D rolls a 5, then die C wins one-third of the time. As students begin to quantify these "if-then" probabilities, it may be helpful to introduce a tree-diagram representation of these conditions, similar to **figure 4**.

With this representation, students begin to develop ideas of conditional probability, such as:

$$\begin{aligned} &P(C>D) \\ &\quad = P(C>D \mid D=1) \times P(D=1) \\ &\quad\quad + P(C>D \mid D=5) \\ &\quad\quad \times P(D=5) \\ &\quad = 1 \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{3} \end{aligned}$$

(The symbol | is read as "given that.")

PREDICT AND EXPLORE

In question 4 of **activity sheet 2**, students review the relationship among the dice examined thus far. Namely,

$$P(A>B) = P(B>C) = P(C>D) = \frac{2}{3}.$$

Die A has an advantage over die B, die B over die C, and die C over die D. Students will now be asked to make predictions about the relationship between die A and die D. Students should not only make conjectures about which die has the advantage but also provide rationales for their hypotheses.

As students discuss their predictions, the teacher may wish to introduce the term *transitivity*. This discussion would include examples of relationships that are transitive, such as *taller than*, *heavier than*, or *more than*. Students may, however, struggle to recognize relationships that are nontransitive. One helpful example is the paper-rock-scissors game: Paper covers rock, rock crushes scissors, yet scissors cut paper.

After this discussion of transitivity, students will test their conjectures to determine the theoretical probability that die D beats die A. In question 6 of **activity sheet 2**, students are asked to represent their findings in two different ways. The teacher should encourage pairs to try one of the strategies presented by their classmates that was not used for the previous comparison.

CONCLUSION

Using Efron's dice is an interesting way to help students explore notions of chance. While throwing these nontraditional dice, they will learn a little more about chance, fairness, and probability.

REFERENCES

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Ben C. Sloop, bsloop@clemson.edu, is a doctoral candidate at Clemson University in Clemson, South Carolina. He enjoys teaching mathematics content courses for future teachers. **S. Megan Che,** sche@clemson.edu, is an assistant professor at Clemson University. She is interested in rational numbers, proportional reasoning, and equity in mathematics and is currently involved in research on gender and mathematics learning.

Nets of Efron's dice are online with Mathematical Explorations at **www.nctm.org/mtms**.

activity sheet 1

Name ______________________________

GAMES WITH DICE A, B, AND C

Students work in pairs, and each player selects either die A or die B. Each player will toss his or her die to see who rolls the higher number.

1. Play the game 10 times. Organize your data in a table.

2. Is this a fair game? Why, or why not?

3. How many times would the pair of dice need to be rolled to determine if the game was fair?

4. From the data you collected, what portion of the time did die A win?

5. Does the outcome of die B affect the result of the game? Why, or why not?

6. Consider only die A. If you were to toss die A six times, how many times would you expect to roll a 4?

7. On average, what fraction of the time would you expect to roll a 4 with die A? Why?

8. In the table below, determine which die would win for each possible outcome.

	A = 0	A = 0	A = 4	A = 4	A = 4	A = 4
B = 3	B					
B = 3						
B = 3						
B = 3						
B = 3						
B = 3						

9. Use the table to find the theoretical probability that die A will beat die B, or $P(A > B)$.

10. Use your answer to question 9 to determine the probability that die A will roll a number smaller than die B, or $P(B > A)$. Explain your reasoning.

Name ______________________________

11. Theoretically, consider the same game with die B rolled with die C. Predict which die has an advantage. Explain your thinking.

12. Discuss your predictions with the class.

For questions 13–18: Students work in pairs, and each player selects either die B or die C.

13. Play the game 10 times. Construct a relative-frequency histogram of your data.

14. Is this game fair? Explain why, or why not.

15. In what fraction of your trials did die B beat die C?

16. What is the theoretical probability that die B rolls a number greater than die C?

17. It is possible that die C will win all 10 times? Is it probable? Why?

18. If you played the game 1000 times, is it possible that die C will win more often than die D? Is it probable? Why?

activity sheet 2

Name ______________________________

GAME WITH DICE C AND D

Students work in pairs, and each player selects either die C or die D. Each player will toss his or her die to see who rolls the higher number.

1. If you could select one of these two dice to play the game as before, which die would you select? Why?

2. What is the probability that die C wins? Organize your evidence below.

3. Compare your solution and strategy with those found by other classmates.

4. Recall the probabilities that you have investigated thus far.

$P(A > B) =$ $P(B > C) =$ $P(C > D) =$

5. Make a prediction about which die you expect to win when die D and die A are rolled. What evidence supports your conjecture?

6. Determine the probability that die D will beat die A. Represent your findings in two different ways.