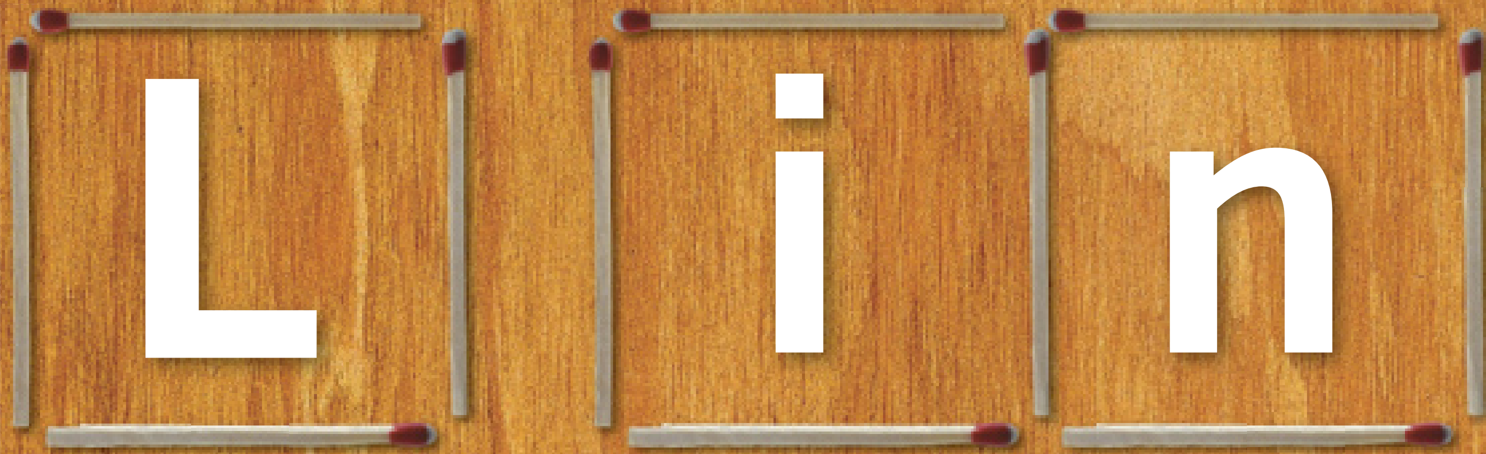
From Arithmetic Lin



Alerting a class to a connection between two familiar topics

Consider the following question, which is typically posed to students who are studying linear equations:

> Suppose a line contains the points (–30, –12) and (10, 48). Find its equation.

The approach often taught to solve such a problem resembles a step-by-step recipe that starts as follows: Step 1: Find the slope using "change in y divided by change in x." Students usually memorize this initial step. If they can find the equation, one wonders whether they understand the underlying concepts.

What follows is an approach to teaching *linear equations* that is based on students' understanding of *arithmetic sequences*. Although these two concepts are closely related, from our experience we have learned that students find arithmetic sequences more intuitive and easier to grasp. The first part of the article focuses on deriving the essential properties of arithmetic sequences by appealing to students' sense making and reasoning. The second part describes how to guide students to translate their knowledge of arithmetic sequences into an understanding of linear equations.

Ryota Matsuura originally wrote these lessons for his mathematics course for preservice elementary teachers. Patrick Harless used the lessons and the described approach with his eighth-grade algebra students and experienced success. The anecdotes drawn from Harless's experience provide insight into how teachers can implement these ideas and how his students made sense of arithmetic sequences and linear equations.

Two forty-minute periods were needed to introduce arithmetic sequences. A third forty-minute period was necessary to translate what students learned about arithmetic sequences into an understanding of linear equations. (For complete lesson plans, as well as the accompanying problem sets, contact Matsuura at matsuura@stolaf.edu.)

ARITHMETIC SEQUENCE: A FIGURAL REPRESENTATION

The goal of this lesson was for students to gain familiarity with arithmetic sequences. The lesson began with the example in **figure 1a**. After reviewing the answer, the teacher introduced the notation b_n (read "b sub n") to denote the nth box number. For instance, we write $b_3 = 10$, because the third box has 10 segments.

To acquire more experience with this sequence and to practice inter-

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Sequences to

Equations

Ryota Matsuura and
Patrick Harless

will help students become more agile with each of them.

preting the new notation, students worked on follow-up questions that were posed using subscripts (see **fig. 1b**). Nearly all students drew figures for the first several box numbers. From their numerical data, they readily identified the "+ 3" pattern in the sequence and explained:

> Each figure adds a square. Because 1 line is already connected, you only need 3 to make another square: 1 on the top, 1 on the bottom, and 1 on the right.

> We add 1 more box, so that's 4 more, but then subtract 1 because of the overlap.

Most students attempted to write equations of the form $y = 3x + \square$ or $y = x + 3$. Those who initially said $y = x + 3$ were trying to capture the "+ 3" pattern but quickly realized that the equation did not yield the desired increase. Another student who first tried $y = 3x$, recognizing that the repeated addition would be captured in multiplication by 3, made this comment:

> When I put 1 into $3x$, it gave me 3; and when I put in 2, it gave me 6. So I knew I had to add 1 [because the first two box numbers are 4 and 7].

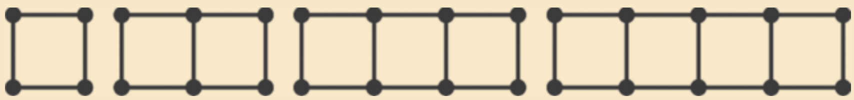
The student thus reasoned that the equation must be $y = 3x + 1$.

When reviewing the answer, the teacher noticed that almost all students had found a correct equation either on their own or with the help of classmates. Since the subscript notation was new, students had written their equation as $y = 3x + 1$ instead of $b_n = 3n + 1$. They also needed frequent reminders about how to interpret expressions such as b_n. The notation did become more comfortable with use, and students learned to appreciate the fact that it helped them distinguish between arithmetic sequences and linear equations.

Some students may have difficulty making the connection between repeated addition (or subtraction) in an arithmetic sequence and its formal representation as multiplication. To probe students' understanding, a teacher might ask those who have made this connection to explicitly discuss why the "+ 3" pattern is being represented as 3 times the number of boxes. ("When you use $b_n = 3n + 1$, it's like starting at 0. When you find the fourth box, it is 0 to 4. That's 4 increases of 3, so you multiply 4 by 3. You need to multiply to see the pattern.") Other students can be asked to interpret and restate in their own words the connection between the two:

Fig. 1 An introductory problem gave students an opportunity to become familiar with subscript notation.

Question: The box numbers follow the figural pattern shown below. Each box number counts the number of line segments used to create the figure.



How many segments are in each of the first 4 box numbers?

(a)

1. Find b_5, the fifth box number.
2. Find b_{10}.
3. Find a formula for b_n, the nth box number.

(b)

Fig. 2 Students were asked to derive a formula for an arithmetic sequence from two given terms.

Suppose you have an arithmetic sequence

$$a_1, a_2, a_3, a_4, a_5, \ldots,$$

with $a_2 = 11$ and $a_5 = 23$.

a. Fill in the table below.

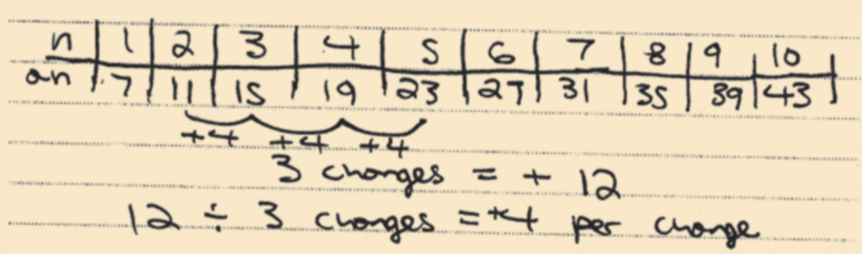
n	1	2	3	4	5	6	7	…
a_n		11			23			…

b. What is the constant difference of this arithmetic sequence? Describe how you found it.

c. If the term a_0 were to exist, what would it be?

d. Find a formula for a_n, the nth term of this arithmetic sequence.

Fig. 3 This student's work reflected a line of reasoning, including the increase of 12 for each three steps.



> We're adding 3 to each box. From the beginning, that is 4 increases [for the fourth box], so it's 4 times 3.

Such a discussion may help those who are still grappling with the connection to begin developing a more mature understanding.

An unexpected discussion occurred as to whether $b_n = 3n + 1$ and $b_n = 3(n - 1) + 4$ could both be correct equations. Student justifications for $b_n = 3n + 1$ have been shown above. Those who wrote $b_n = 3(n - 1) + 4$ explained, "The first box number is 4. Then you add 3 every time beginning with the second, so you have to subtract 1 [from n]." One student suggested $b_n = 4n - (n - 1)$ and reasoned:

> $4n$ is the number of the sides of squares if you don't connect them. But if you do, there are segments that are connected and counted twice. If there are 2 squares, there is 1 connected line; if there are 5 squares, there are 4 connected lines; if there are n squares, there are $n - 1$ connected lines, so that's what you subtract.

Students eventually agreed that all three representations were equivalent. Their flexible thinking about arithmetic sequences led to their natural inclination to simplify these algebraic expressions and establish their equivalence. Harless explained that the box-number sequence is an example of a general way to express a pattern. He then defined these terms:

- *Arithmetic sequence*—Any sequence with a constant difference between the terms.
- *Constant difference*—The fixed number that is added.

For example, the box numbers form an arithmetic sequence with a constant difference of 3. Depending on their algebra background, students may need to work with more examples of arithmetic sequences (in particular, using the new notation and generalizing to equations involving n) before moving on.

ARITHMETIC SEQUENCE: GIVEN TWO TERMS

In the next lesson, students were asked to derive a formula for an arithmetic sequence, including finding the constant difference if given two terms of the sequence. (See **fig. 2**.) After completing the table, most students claimed that the constant difference was "obviously" 4: "From a_2 to a_5, the sequence went up by 12. That took 3 steps, so it's an increase of 4 every step." (See **fig. 3**.) After this discussion, the teacher summarized their work by writing the following:

$$\text{constant difference} = \frac{\text{increase in sequence values}}{\text{the number of steps it takes}} = \frac{23-11}{5-2} = \frac{12}{3} = 4$$

To find a_0, many students subtracted the constant difference of 4 from $a_1 = 7$ and obtained $a_0 = 3$. Kevin also suggested using the given term $a_2 = 11$ and subtracting the constant difference *twice* so that $a_0 = 11 - (2)4 = 3$.

By now, most students understood that a formula for this sequence would have the form $a_n = 4n + \square$. Eventually, Kevin derived the formula $a_n = 3 + 4n$. After seeing Kevin's formula on the board, several others were able to explain that to find a_n, they would start at $a_0 = 3$ and take n steps forward, where each step is a constant difference of 4.

When asked to find the general form of the nth term, some students realized right away that a_0 would play a role in the formula. However, several others computed a_0 as if it were an unrelated problem, then found the missing constant term of $a_n = 4n + \square$ through experimentation. Harless heard these comments: "Oh, it's just a_0!" or "So *that's* why they asked me to find a_0." These students then explained that "a_0 is what you have before you start adding the 4s."

At this point, students had essentially derived the slope formula on their own. They had also found an equation of a line, given its two points. As you will soon see, all that remained was to translate the results about arithmetic sequences into the language of linear equations.

Later on, students encountered the problem in **figure 4**, which asked them to derive the general formula for arithmetic sequences. Most students were comfortable interpreting the given formula, as they (like Lena) did not use the table to answer questions (b) and (c). They explained, "For the constant difference, 7 is multiplied by n, which means that everything is a difference of 7" and "Lena means that by looking at the equation, the constant, which is –3, is the answer for d_0, and the 7 tells her the constant difference." Finally, a student summarized the discussion on the board, as shown in **figure 5**, which prompted students to make the following generalization:

> A formula for an arithmetic sequence has the form
>
> $$a_n = a_0 + (\text{constant difference}) \bullet n.$$

But more important than the formula itself was the fact that students derived it by themselves and thus truly understood the concepts it represented.

Fig. 4 Questions about a specific arithmetic series helped students derive the general formula.

Suppose you have an arithmetic sequence

$$d_1, d_2, d_3, d_4, d_5, \ldots,$$

where each term is given by the formula $d_n = -3 + 7n$.

a. Fill in the table below.

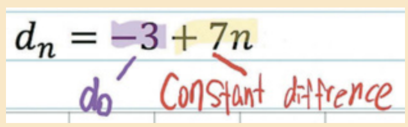
n	1	2	3	4	5	6	7	…
d_n								…

b. What is the constant difference of this arithmetic sequence?

c. If the term d_0 were to exist, what would it be?

d. Lena says, "I could have answered questions (b) and (c) *without* answering question (a) first." What does she mean? Explain.

Fig. 5 This symbolic representation prompted students to make a meaningful generalization about linear functions.



CONNECTION TO LINEAR EQUATIONS

Students translated their experiences with arithmetic sequences into a genuine understanding of linear equations. This lesson began with the question in **figure 6a**. Students had trouble getting started on this exercise, even though they were capable of plotting points, because the notation (n, b_n) confused them. Harless listed a few points that would be on the graph. For example, the box number $b_3 = 10$ corresponds to the point (3, 10) on the coordinate plane. Students began to see what was being asked and could complete the graph without further difficulty. One student sketched the graph shown in **figure 6b** on the board. Notice how she first plotted the points (n, b_n) from the box-number sequence and then drew the line passing through these points.

Students saw that these points appeared to form a line, which was what they had expected. When asked why they anticipated a line, they responded, "Because they increase by the same amount each time." When asked if this would be true for *any* arithmetic sequence, they said, "Yes, because every arithmetic sequence goes up by the same amount each time."

Then Harless asked whether or not $b_n = 1 + 3n$ is an equation of this line. He reminded students that $b_n = 1 + 3n$ is an equation for the box-number sequence from their first exercise and asked, "What does n represent?" ("It's the number of boxes.") "What would happen if we let $n = 2.4$?" ("We wouldn't have a meaningful box number.") He said that on the original graph with the points plotted, only the box numbers are represented; the values of n are the "counting numbers." But on the graph of the line that connects the points, $n = 2.4$ is included even though $b_{2.4}$ has no meaning as a box number. Therefore, the graph of $b_n = 1 + 3n$ is given by the discrete points (n, b_n) only and not by the line passing through these points.

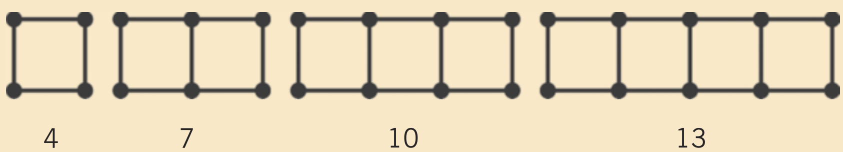
The teacher explained that to account for *all* points on the line (not just those from the box-number sequence), they must replace n and b_n with x and y. Thus, an equation of the line is $y = 1 + 3x$. To verify this claim, suppose $x = 2.4$. Then,

$$y = 1 + 3(2.4) = 8.2.$$

Students could see from the graph that the point (2.4, 8.2) is, indeed, on the line. They experimented with a few other noninteger values of x. The teacher then said that an equation such as $y = 1 + 3x$ can be used as a *point tester* (i.e., to test whether any point on the coordinate plane is on the graph of the equation) (EDC 2009). Given this correspondence between an arithmetic sequence (e.g., $b_n = 1 + 3n$) and a linear equation (e.g., $y = 1 + 3x$), Harless defined the slope of a line as the constant difference of its corresponding arithmetic sequence. For example, the line corresponding to the box-number sequence has slope 3, because the sequence has a constant difference of 3.

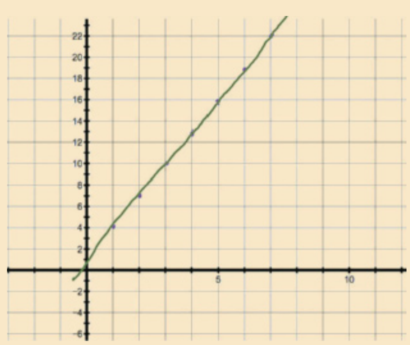
Fig. 6 A knowledge of linear equations and subscript notation was required to complete this task.

Recall the box number pattern:



Make a graph of the points (n, b_n), where b_n represents the nth box number.

(a)



(b)

Next, students delved into pairs of problems like those in **figure 7**. They were asked equivalent questions in two different ways, first in the language of arithmetic sequences and then in the language of linear equations.

Problem 1 was a familiar question, and most students found the formula

$$c_n = 5 + \frac{2}{3}n$$

easily. But some students, despite having a strong understanding of arithmetic sequences, struggled with the fact that the constant difference was a fraction. They converted it to a decimal approximation, as shown by the following response:

> The constant difference is 0.67. To find this, I found the difference between 21 and 31 and divided that by

the difference between the subscripts (24 and 39). That equaled 0.67, so the constant difference is 0.67.

Rounding led to incorrect answers (for c_0 and c_n), but also gave the teacher an opportunity to discuss the merits of using exact values. Problem 2a went smoothly, with several students gleefully declaring, "It's the same problem!" They realized that the points (24, 21) and (39, 31) on the line corresponded to the terms $c_{24} = 21$ and $c_{39} = 31$ of the arithmetic sequence. Thus, students used their results from problem 1 to immediately conclude that the line has slope 2/3 and the equation

$$y = 5 + \frac{2}{3}x.$$

Problem 2b of **figure 7** prompted students to make the connection between the start value $c_0 = 5$ of the arithmetic sequence and the y-intercept (0, 5) of the corresponding line. The teacher did not yet introduce the term y-intercept in this lesson. Students readily recognized that "the line crosses the y-axis at 5. It's the value of c_0." Thus, when later introducing the formal term of y-intercept, students will have a solid understanding of the concept, and the term will be easily connected to already established ideas.

Problem 2c was an application of the point-tester notion. All students were comfortable with it and, aside from those who converted their answers to decimal approximations, had no difficulty with its implementation (see **fig. 8**). Students continued to work on similar pairs of problems. They found the questions about arithmetic sequences more intuitive and easier to understand. Problem 2 is a typical question that students face when studying linear equations, but it required nothing more than a simple translation from the language of arithmetic sequences to that of linear equations.

Fig. 7 Students were asked equivalent questions in two different ways, first in the language of arithmetic sequences and then in the language of linear equations.

1. Consider an arithmetic sequence with $c_{24} = 21$ and $c_{39} = 31$. What is its constant difference? What is c_0? The formula for c_n?

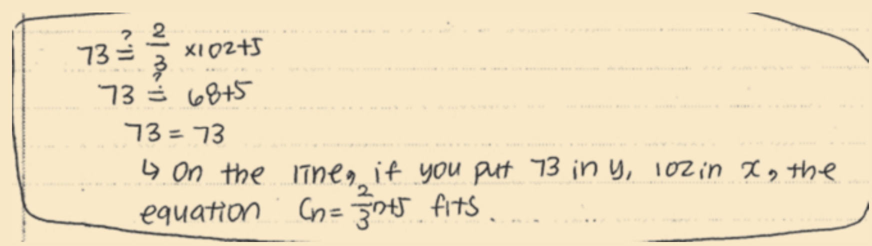
2. Suppose a line contains the points (24, 21) and (39, 31).

a. Find the slope and equation of this line. (*Hint:* This should be easy if you have completed question 1.)

b. Where does the line cross the y-axis? How is this point related to the original arithmetic sequence from problem 1?

c. Is the point (102, 73) on the line? How do you know?

Fig. 8 This student response used the equation that was derived as a point tester.


$73 \overset{?}{=} \frac{2}{3} \times 102 + 5$

$73 \overset{?}{=} 68 + 5$

$73 = 73$

↳ On the line, if you put 73 in y, 102 in x, the equation $C_n = \frac{2}{3}n + 5$ fits.

Fig. 9 Flexibility was added to this problem in that arithmetic sequences allowed n to take on all integer values (including negatives).

Suppose you have an arithmetic sequence

$$\ldots, d_{-3}, d_{-2}, d_{-1}, d_0, d_1, d_2, d_3, \ldots,$$

which extends infinitely in both negative and positive directions. Moreover, suppose that $d_{-6} = 1$ and $d_{18} = 17$. What is its constant difference? What is d_0? The formula for d_n?

Students then worked on the question in **figure 9**, which gave them more flexibility when using arithmetic sequences by allowing n to take on all integer values (including negatives). Students felt comfortable with this problem, and the negative subscript did not present any difficulty. They considered the number of steps between –6 and 18 when finding the constant difference, which was 2/3. To find d_0, most students took 18 steps back from $d_{18} = 17$ by subtracting $2/3 \times 18$ from 17. Finding the formula

$$d_n = 5 + \frac{2}{3}n$$

was routine by this point.

Finally, students were asked:

Suppose a line contains the points (–30, –12) and (10, 48). Find its equation.

Many approached this problem by first translating the given points to terms of an arithmetic sequence. One student commented, "I found this by changing it into a problem [an arithmetic sequences problem] not coordinates. Then I did the same things as the other problems." Several students clearly articulated the correspondence between the line passing through (–30, –12) and (10, 48) and the arithmetic sequence with $a_{-30} = -12$ and $a_{10} = 48$. By this time, none of the arithmetic sequences posed a challenge, and students understood that they could recast the linear equations problem in the language of arithmetic sequences and then apply the familiar techniques (see **fig. 10**).

CONCLUSION

We have described a way of learning linear equations in which students first study arithmetic sequences. Our approach, which relies on the intuitive nature of arithmetic sequences, makes learning linear equations more accessible to students.

We end with a story about Katie, who had a formulaic understanding of linear equations before these lessons were taught. In fact, Katie had written the notes in **figure 11a** at the top of her activity sheet. When asked about an arithmetic sequence with $a_1 = 3$ and $a_4 = 15$, she found the constant difference using an approach that made sense to her (see **fig. 11b**). Next, when asked about a line containing (1, 3) and (4, 15), she computed the slope using the traditional slope formula, then recognized the connection between the two concepts (see **fig. 11c**).

Similar to Katie, several students in Harless's class had a cursory and rote-memorized understanding of linear equations. Such students benefited as much or even more from these lessons than the students for whom the topic was essentially new.

Note: To implement these lessons in your classroom using the activity sheets and the lesson plans, contact the author at matsuura@stolaf.edu.

Fig. 10 Students understood that they could recast the linear equations problem in the language of arithmetic sequences and then apply familiar techniques.

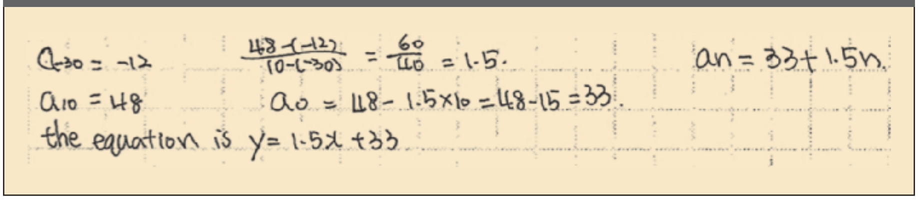
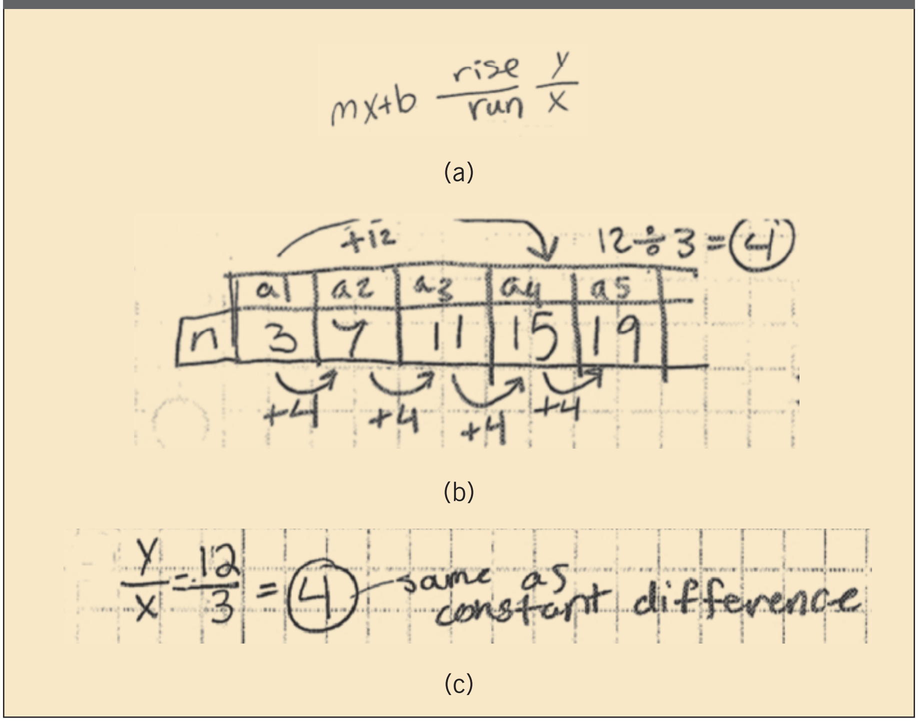


Fig. 11 Learning arithmetic sequences allowed Katie to draw deeper connections, as shown by these explanations.



(a)

(b)

(c)

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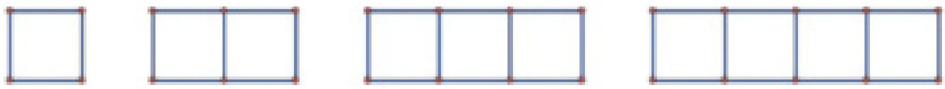
Download one of the free apps for your smartphone. Then scan this tag to access **www.nctm.org/mtms019** to find a continuation of the activity sheet.

activity sheet

Name ______________________________

ARITHMETIC SEQUENCES

1. You have seen that the first four box numbers are 4, 7, 10, and 13, as shown below.



a. Find b_5 (i.e., the fifth box number).

b. Find b_{10}.

c. Find b_{100}.

d. Find b_n (i.e., a formula for the nth box number).

e. Challenge: Is 5000 a box number? Why, or why not?

2. Determine if each sequence below is an arithmetic sequence. Explain how you know.

a. 2, 9, 16, 23, 30, 37, 44, 51, . . .

b. 1, 2, 3, 4, 5, 6, 7, 8, . . .

c. 3, 7, 13, 21, 31, 43, 57, 73, . . .

d. 7, 7, 7, 7, 7, 7, 7, 7, . . .

e. 25, 19, 13, 7, 1, −5, −11, −17, . . .

Name ______________________________

3. Suppose you have an arithmetic sequence

$a_1, a_2, a_3, a_4, a_5, \ldots$

with the following properties:

- $a_2 = 11$ (i.e., the second term is 11)
- $a_5 = 23$

a. Fill in the table.

n	1	2	3	4	5	6	7	. . .
a_n		11			23			. . .

b. What is the constant difference of this arithmetic sequence? Describe how you found it.

c. If the term a_0 were to exist, what would it be?

d. Find a_{100}.

e. Find a_n (i.e., a formula for the *n*th term of this arithmetic sequence).

4. Suppose you have an arithmetic sequence

$c_1, c_2, c_3, c_4, c_5, \ldots$

with the following properties:

- $c_{20} = 99$
- $c_{31} = 154$

a. What is the constant difference of this arithmetic sequence? Describe how you found it.

b. If the term c_0 were to exist, what would it be?

c. Find c_{100}.

d. Find c_n.

e. If $c_n = 359$, find n. What does your answer mean?

A continuation of the activity sheet is online at ***www.nctm.org/mtms019****.*