Get to Know a Ci

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While handling Play-Doh® and working within the levels of the van Hiele framework, students write and critique definitions of a circle.

Consider a teaching trajectory that includes modeling to support the act of defining a circle: In kindergarten, students are typically able to identify a circle from other items on the basis of earlier experiences with round objects; from kindergarten through grade 8, students progress to learning definitions (radius and diameter) as well as formulas for circumference and area. However, when does the formal definition of a circle become important and accessible? *Principles and Standards for School Mathematics* states that in grades 6–8 all students should "analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships" (NCTM 2000, p. 232).

An examination of reform curricula indicates that students should be given ample opportunity to describe and classify the properties of two- and three-dimensional shapes. The act of sorting shapes by looking at different attributes is an excellent way to differentiate between classes of objects, which helps define the shapes. For example, sorting triangles by angle measure results in three classifications: acute, right, and obtuse. Sorting triangles by side length results in scalene, isosceles, and equilateral categories. An interesting follow-up activity would be to decide if one could draw an obtuse scalene triangle or a right equilateral triangle (yes and no, respectively). We, the authors, conjecture that the act of identifying properties of two- and three-dimensional objects falls short of what middle-grades students are able to accomplish with regard to creating or verifying valid definitions.

ALLOY PHOTOGRAPHY/VEER

Two basic requirements are necessary to understand a definition. One must—

1. have a concept image of the object that is being defined; and
2. understand what it means to engage in the act of defining (Zandieh and Rasmussen 2010).

Middle-grades students have a concept image of a circle, but they lack experience in the act of defining. The structure of definitions needs to include conditions that are necessary and sufficient. Most students are able to identify necessary conditions, but they have trouble determining if sufficient conditions are met. For example, it is necessary to say that a square is "a polygon with four equal sides and four angles of 90 degrees" (Zazkis and Leikin 2008, p. 135). However, this statement is not sufficient, since polygons are not limited to quadrilaterals. For example, it is possible to draw a hexagon with four equal sides and four right angles that satisfies the necessary conditions but is clearly not a square.

The question then becomes this: How do we engage students in the act of defining, which involves learning the characteristics of a definition? One answer might be to look at a teaching trajectory guided by the van Hiele levels of thought (Hoffer 1983). This teaching trajectory scaffolds the process with specific lesson phases: Inquiry, Directed Orientation, Expliciting, Free Orientation, and Integration.

Before reading further, write the definition of a circle. In our experience, learners of all ages typically define a circle by listing traits or properties (that sometimes create something other than a circle).

Sara Paul (the second author) taught three classes of sixth-grade mathematics. What follows is a description of what occurred when Paul teamed with a math educator and conducted an activity to develop the definition of a circle. What occurred in the three classes was fairly similar, so we combined the actions in the text that follows. A flip camera was used throughout the day to record the class discussion and student work. The description of the circle activity is presented within the context of the van Hiele levels of thought.

INQUIRY LAUNCH

To begin, students were asked to write their definition of a circle and to share their definitions with others in their small group. As we walked around the room, we observed the students' writing. Then we discussed the definitions as a class. As students volunteered their definitions, we recorded the key points on the board and concluded with a list of responses. The definitions generated by the students can be categorized within the van Hiele levels of geometric reasoning. Students operating at Level 0: Visualization (V-0) focus on the appearance of the shape, or what it "looks like"; students at Level 1: Analysis (V-1) refer to properties of an object. This list is a synthesis of what was recorded in each of the three classes.

V-0 Level: Appearance

- Round shape
- Looks like a coin or orange
- Has no corners and goes forever

V-1 Level: Properties

- No corners, sides, angles, or vertices
- 360 degrees
- Infinite lines of symmetry
- Contains a diameter and radius
- All diameters are the same length

The first three V-0 responses use everyday language and are based on the students' experiences with round objects. The last five V-1 responses refer to properties of a circle. The reference to a circle as "360 degrees" is a definition we have recorded every time we have used this activity, but it was particularly problematic with this group of students.

A student named Rebecca (all names are pseudonyms) stated that "a circle is a shape that is a 360 degree angle with no vertices and is round." Many of the students had this definition written on their papers. Paul explained that her students were recalling from their fifth-grade math class that a circle is a 360 degree angle. She illustrated the 360 degree angle by drawing the sketch in **figure 1**. We will return to this misconception in the Free Orientation phase of the activity.

After all definitions were shared, we investigated their accuracy. We wanted students to test each definition to see if it was possible to draw a shape other than a circle. The students quickly decided the first two definitions (round shape and looks like a coin or orange) could apply to several shapes that might not be circles, so we crossed through them on the board.

As we continued to study the list, we demonstrated how to test a definition by trying to draw a shape on the board that adhered to the definition but was not a circle. For example, we drew a closed curve that had no corners, sides, angles, or vertices (shown in **fig. 2a**), which was clearly not a circle. The students were hooked. The resulting discussion generated a lot of

Fig. 1 This 360 degree angle was sketched to represent a circle definition.

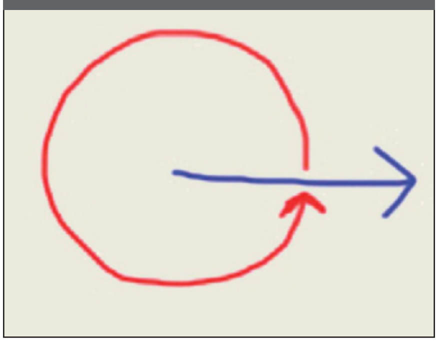


Fig. 2 To test students' definitions of a circle, shapes were drawn that adhered to those definitions.

(a) No corners, sides, angles, or vertices

(b) Curved line with no opening

(c) Contains a 360 degree rotation

interest as volunteers asked to come to the board and draw pictures that would test the remaining "definitions."

When Josh read the definition, "A circle has no corners and goes forever," Morgan drew a line segment. We moved to the next definition, and Josh read, "A circle has curved lines and no opening, a circle is 360 degrees." Morgan erased the segment and drew **figure 2b** to show a curved line with no openings. We offered assistance with the 360 part of the definition by drawing a spiral shape on the board (see **fig. 2c**).

When we looked at the last three definitions on the list (infinite lines of symmetry, contains a diameter and a radius, and all diameters are the same length), students noted that these were true statements about circles, but they did not help us actually draw a circle. Instead of crossing them off the list, we asked the students to label them as "properties of a circle." At this point, it was clear that the students were unable to use any of their definitions to draw a circle exclusively. To develop a more accurate definition, students explored the creation of a circle through Play-Doh®.

DIRECTED ORIENTATION PHASE

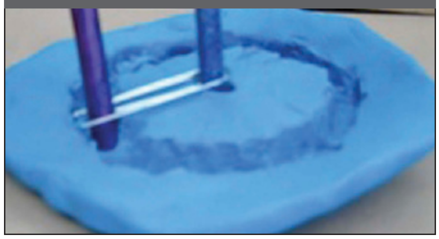
The students worked with a partner at their desks to create a model of a circle using two pencils, a large wire paper clip, and a can of Play-Doh. We directed them through the following three steps:

1. Flatten the Play-Doh on your desk or table.
2. Place one end of the first pencil in the middle of the Play-Doh and slide the paper clip over the pencil.
3. Secure the second pencil inside the other end of the paper clip and poke a hole into the Play-Doh. Lift the second pencil to a new spot and poke another hole; do not drag the pencil in the Play-Doh. Be sure to keep the pencils at each end of the paper clip. (As more holes are made, the circle should become apparent.)

An example of a student-produced circle is shown in **figure 3**. Students were not directly told that they would be creating a circle during this activity. The goal was that by creating many points equidistant from a given point, they would be able to focus on the structure of this shape.

As each step of the activity was presented, we monitored student work. A few students needed to be reminded to continue making holes, and others had to be encouraged to poke holes instead of dragging the pencil in a circular motion. The most important element of the Play-Doh activity was for the students to verbalize the connection between the construction process and the definition of a circle. This was achieved through four discussion questions we posed in the Expliciting phase of the activity.

Fig. 3 Students generated points equidistant from one specific point to produce a Play-Doh circle.



EXPLICITING PHASE

The transition between these teaching phases was subtle and occurred as students were taking turns poking holes in their flattened pieces of Play-Doh. While we circulated around the classroom, we found ourselves asking the students the following four questions to guide their thinking. Students needed prompting to synthesize what the hands-on activity was showing them. These questions were discussed as a whole class, after everyone had finished the activity. The questions focused on the structure of the shapes they had created and culminated in a refined definition.

Question 1. How many points are there?
As teachers, we made a transition from talking about holes in the Play-Doh to talking about points on the circles. The students easily adjusted to the more formal mathematics vocabulary as we explored the number of points required for the definition of a circle. Even though the students knew they had made lots of points in the Play-Doh, it was unclear to them whether to use "a set of points" or "an infinite set of points" in their definition. This issue was anticipated, so Paul asked, "How many points are in a line?" She also mentioned to the class that between any two points is another point. "A set of points" could refer to a very specific quantity of points, so students decided to use "an infinite set of points" in their definition of a circle.

Question 2. What are the roles of the pencils and paper clip?
Students observed that all the points were created by moving one pencil around the other pencil, connected by a paper clip. These points were the same distance from the first pencil, in the middle of the Play-Doh. They knew that this pencil's position was called the *center of the circle*. Students also knew what a radius was, but they did not recognize it as being an important part of the circle's construction. They were more concerned with the diameter and not the radius. It took a bit of prompting to get even one student to say that the length of the paper clip determined where the points would go. At that time, we introduced the term *equidistant*. Although we did not mention that we have a tool that can be used to construct a circle, a few students made this connection. Chris stated, "OK, it was like a compass because you put one pencil in the middle with the paper clip and then kept putting the other pencil around and rotating it."

Question 3. Should the points be connected?
A number of students stated that a circle is the line formed by connecting an infinite number of points. This part of their definition was also expected because when students work with points, they typically connect them. The physical act of "connecting" had to be addressed. We asked the students, "What would happen if you continued to poke holes [make points] in the Play-Doh for the next few hours or until the next day? Would you need to 'connect' anything?" After thinking about these two questions and discussing in their small groups, they concluded that the points were so close together that they appeared to be connected, when in fact, they were not.

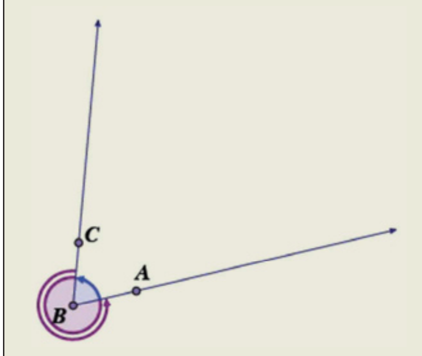
Question 4. What is the definition of a circle?
At this point, the students had refined their definition of a circle to be "an infinite set of points equidistant from a given point." To finish the activity, we needed to test their refined definition, so we asked the students to pretend that the center of a circle was on their shoulder and that one point was the tip of their finger with the arm fully extended. Could they make a shape that satisfied the definition but was not a circle? Several said that it would make a sphere. This prompted a discussion of the difference among one, two, and three dimensions. The students explained that three-dimensional shapes have length, width, and height, so the circle had to be a two-dimensional shape. Our students' final definition was this: "A circle is a two-dimensional shape with an infinite set of points equidistant from a given point."

FREE ORIENTATION AND INTEGRATION PHASES

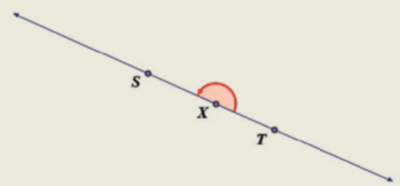
After returning Play-Doh to the cans and cleaning up, we revisited the definition of the circle as a 360 degree angle. First, we asked the students to draw a 0 degree angle, which prompted a discussion on the definition of an angle (two distinct rays with a common endpoint). Then, we drew a few angles on the board and discussed the role of arc marks. Single and double arc marks were used in angle *ABC* shown in **figure 4a** to distinguish between the interior and exterior. The straight angle *SXT* in **figure 4b** is labeled with a semicircular mark. The students agreed that the definition of a semicircle is not "a 180 degree angle."

We also pursued a discussion regarding how a circle resembled a rounded square, an oval, and a rounded rectangle. Paul drew diagonals in the square and the major and minor axes in the oval to provide vocabulary for the discussion. These actions supported the connections that the students were making as they compared properties of the shapes.

Fig. 4 Angle tic marks identifying the interior and exterior helped students determine that "180 degrees" is not a sufficient definition for a semicircle.



(a)



(b)

THE TRAJECTORY CONCLUDED

The teaching trajectory completed with the sixth graders progressed from using informal language, to moving through a hands-on activity and discussion, and concluding with a formal definition. Van de Walle emphasizes the importance of geometric experiences for fostering student growth:

> Students should explore, talk about, and interact with content at the next level, while increasing their experiences at their current level. . . . All teachers should be aware that the experiences they provide are the single most important factor in moving children up this developmental ladder. (2010, p. 404)

We encourage teachers to apply the van Hiele teaching trajectory beyond this circle activity. For example, using four straws of equal length, connected end to end with yarn, can help students learn to develop the definition of a rhombus (the square is a special case). As they learn about functions, the hands-on representations of function machines and T charts are useful tools to help students progress toward using formal language and symbols.

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