
SWEET

The Maximum Chocolate Party game requires students to divide and compare fractions in a practical and concrete context.

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Almost everyone loves chocolate. However, the same cannot be said about fractions, which are loved by markedly fewer. Middle school students tend to view them with wary respect, but little affection. We attempt to sweeten the subject by describing a type of game involving division of chocolate bars. The activity we describe provides a context for ordering and manipulating fractions. The use of a number line as tool for understanding fractions will be discussed. This activity not only encourages students to think deeper about fractions but also highlights students' ability to generalize and reason—crucial skills in learning mathematics. It may serve to lead students into higher-order tasks involving the formulation and exploration of hypotheses about numbers and operations.

We present a few variations of the Maximum Chocolate Party game; teachers can easily create others. This game has been used repeatedly in professional development seminars for middle school teachers. The discussions recorded here have been part of those seminars. This material can be used with little or no variation with middle school students.

THE GAME

There are three tables at the Maximum Chocolate Party (MCP). One, two, and three chocolate bars, identical in size and flavor, are on the first, second, and third tables, respectively. Each guest, on arrival, chooses a table and sits down. The rules of the MCP state that when all guests are seated, the chocolate from each table is divided equally among those at the table. For the purpose of this game, we assume that all players want to maximize their share of chocolate and that every guest chooses as if he or she were to be the last.

PRACTICAL MOMENTS

When playing this game in the classroom, place identical pieces of paper on the tables, as shown on the **activity sheet**. Participants may stand a few feet away from the tables. They should be able to see the "chocolate" on every table well, but at the same time, they

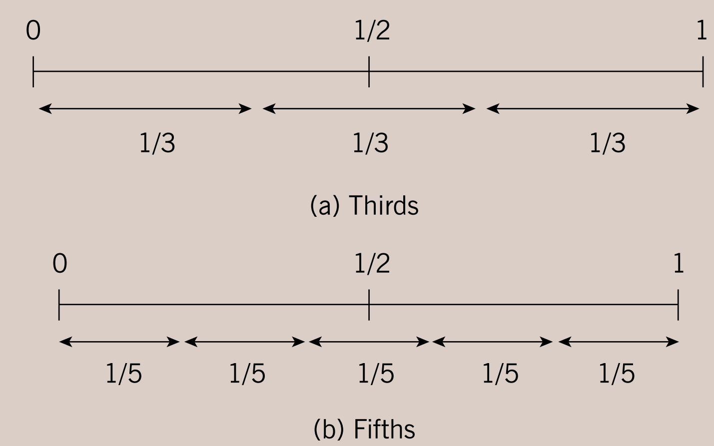
WORK WITH FRACTIONS

(participants who are still standing) should not be mistaken for those already "sitting at the tables." Every participant should have a copy of the **activity sheet**. After every new guest is seated, all participants (including those already sitting) should take a few minutes to decide where the next guest should go. It is important to ask every new guest to explain clearly why he or she chose a particular table. All others should listen carefully to be able to agree or disagree and be ready for discussion. Before playing the game, review strategies for comparing fractions.

Table 1 The grid shows how much chocolate every new guest would receive at each table. The unshaded cell shows the table chosen by this guest.

Guest Number	The First Table (1 Chocolate Bar)	The Second Table (2 Chocolate Bars)	The Third Table (3 Chocolate Bars)
1	1	2	3
2	1	2	1 1/2
3	1	1	1 1/2
4	1	1	1
5	1/2	1	1
6	1/2	1	3/4
7	1/2	2/3	3/4
8	1/2	2/3	3/5
9	1/2	2/4	3/5
10	1/2	2/4	3/6
11	1/3	2/4	3/6
12	1/3	2/4	3/7
13	1/3	2/5	3/7
14	1/3	2/5	3/8
15	1/3	2/6	3/8
16	1/3	2/6	3/9

Fig. 1 These two divisions of a number line from 0 to 1 show that 2/3 is greater than 3/5.



STEP-BY-STEP DISCUSSION DURING THE GAME

See **table 1** to follow the description. Because the chocolate bars are identical, we will use the term *bar* as a unit throughout this article. The first guest should go to the third table containing 3 chocolate bars. The second guest would get 1 chocolate bar at the first table, 2 bars at the second table, and 1 1/2 bars at the third table because the 3 bars are shared. The second guest sits at the second table. The third guest would get 1 bar at the first table, 1 bar at the second table, and 1 1/2 bars at the third table, so the third guest sits at the third table.

Guests 4, 5, and 6

The fourth guest can go to any table, because exactly 1 chocolate bar awaits him or her at all tables. This may be a good place to pause and have students convince themselves that in the case of a three-way tie, the next three guests will choose tables one, two, and three in *some* order. Regardless of their choices, the next guest in line will face exactly the same situation. For example, in **table 1** the fourth, fifth, and sixth guests happened to choose tables one, three, and two. The reader can verify that if the fourth, fifth, and six guests had ordered their table choices differently, the choices of the seventh guest would be exactly the same.

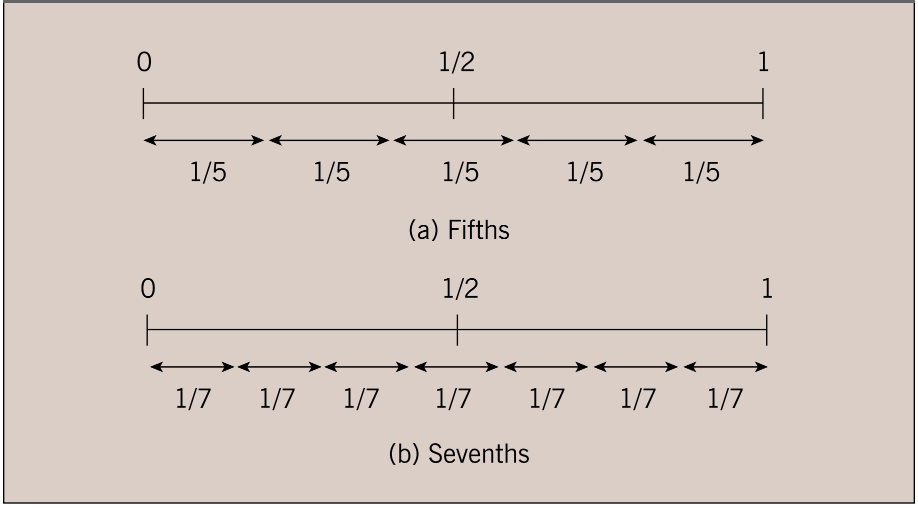
Guest 7

The decision of the next guest deserves a separate story. The seventh guest faces 1/2 bar at the first table, 2/3 bar at the second table, and 3/4 bar at the third table. Which is the greatest amount: 1/2, 2/3, or 3/4? We can think about every one of these three fractions as "one unit fraction less than the whole." However, the units are different: 1/2 is 1/2 less than the whole, 2/3 is 1/3 less than the whole, and 3/4 is 1/4 less than the whole. Thus, we may start by comparing the unit fractions 1/2, 1/3, and 1/4.

Imagine 3 identical chocolate bars. One bar is divided into two equal parts, 1 bar into three equal parts, and 1 bar into four equal parts. Which parts are the smallest, and which are the greatest? The more parts you have, the smaller they are. Informally stated: "It is wonderful to have a lot of friends, but *not* when you are sharing chocolate." After hearing this, students laugh, but then everybody confidently agrees that 1/2 > 1/3 > 1/4. Mathematically, it can be stated as follows: Unit fractions can be compared by comparing denominators—the greater the denominator, the smaller the value of the unit fraction.

Imagine three unwrapped identical chocolate bars. One is missing 1/2, one is missing 1/3, and one is missing 1/4. In which bar does more chocolate remain? The answer is the same as the answer to "where less is missing." When the second question is answered, students can conclude that 1/2 < 2/3 < 3/4. This means that the seventh guest should go to the third table.

Fig. 2 These two divisions of a number line from 0 to 1 demonstrate that 2/5 is less than 3/7.



Guests 8 through 12

The eighth guest will have to compare 2/3 and 3/5 because they both are greater than 1/2. But *how much* greater? **Figure 1** shows that 2/3 is greater than 1/2 by half of 1/3. At the same time, 3/5 is greater than 1/2 by half of 1/5. However, 1/5 is smaller than 1/3. Thus, half of 1/5 is less than half of 1/3. So the second table should be chosen. The ninth guest will have to go to the third table.

The tenth guest can choose any table, as he or she would get the same amount of chocolate, 1/2 of a bar. However, at the second table, this amount is presented as 2/4 and at the third table as 3/6. This situation provides a good opportunity to discuss equivalent fractions. Let's say the tenth guest chose the first table and the eleventh guest chose the third table. Then the twelfth guest will have to choose the second table.

Guest 13

Which table should be chosen by the thirteenth guest? Let's start by comparing 1/3 and 2/5. Using equivalent fractions, 1/3 = 2/6. However, 2/5 is greater than 2/6. Previously, we have established that, for unit fractions, the greater the denominator, the smaller the value of the fraction. Thus, 1/5 is greater than 1/6. We have two of each such parts. As a result, 2/5 > 2/6. Thus, the rule works not only for unit fractions but also for any fractions with common numerators. (Imagine placing an equal number of weights on both sides of a scale. All the weights on the right side are equal among themselves, and all the weights on the left side are equal among themselves. But every weight on the left is lighter than every weight on the right. Which combined quantity is lighter?)

Now we need to compare 2/5 and 3/7. They are both less than 1/2. An illustration shown in **figure 2** is similar to **figure 1** and can demonstrate that 2/5 is less than 1/2 by half of 1/5 and that 3/7 is less than 1/2 by half of 1/7. Because 1/5 is greater than 1/7, 2/5 is farther to the left on the number line from 1/2 than 3/7. This shows that 2/5 < 3/7, and the thirteenth guest should go to the third table.

And More Guests

The fourteenth guest may start by saying that it has already been shown that 1/3 = 2/6 < 2/5. Following this logic, 1/3 = 3/9 < 3/8. So we need to compare 2/5 and 3/8. As was previously stated, 2/5 is less than 1/2 by half of 1/5. In other words, 2/5 is less than 1/2 by 1/10, or

$$\frac{2}{5}=\frac{1}{2}-\frac{1}{10}.$$

At the same time 1/2 = 2/8, and 3/8 is less than 1/2 by 1/8, or

$$\frac{3}{8}=\frac{1}{2}-\frac{1}{8}.$$

Again, as with the seventh guest, we are subtracting from the same amount, but 1/10 is less than 1/8. Thus, 2/5 is greater than 3/8, and the second table should be chosen. The fifteenth guest should choose the third table. The sixteenth guest would get an amount equal to 1/3 of a chocolate bar at every table.

This seems like a convenient place to stop, even though theoretically this game can be continued for an unlimited number of guests.

A FEW MORE WORDS ABOUT STRATEGIES

While playing this game, away from desks and calculators, it seems natural to avoid translating fractions into decimal forms or using common denominators for comparison.

Table 2 This grid shows how much chocolate every new guest would have received at each of three tables, one with 1 chocolate bar, one with 3 bars, and one with 4 bars to share. The unshaded cell shows the table that was chosen by this guest.

Number of the Guest	The First Table (1 Chocolate Bar)	The Second Table (3 Chocolate Bars)	The Third Table (4 Chocolate Bars)
1	1	3	4
2	1	3	2
3	1	1 1/2	2
4	1	1 1/2	4/3
5	1	1	4/3
6	1	1	1
7	1/2	1	1
8	1/2	1	4/5
9	1/2	3/4	4/5
10	1/2	3/4	4/6
11	1/2	3/5	4/6
12	1/2	3/5	4/7
13	1/2	3/6	4/7
14	1/2	3/6	4/8
15	1/3	3/6	4/8
16	1/3	3/6	4/9
17	1/3	3/7	4/9
18	1/3	3/7	4/10
19	1/3	3/8	4/10
20	1/3	3/8	4/11
21	1/3	3/9	4/11
22	1/3	3/9	4/12
23	1/4	3/9	4/12
24	1/4	3/9	4/13
25	1/4	3/10	4/13
26	1/4	3/10	4/14
27	1/4	3/11	4/14
28	1/4	3/11	4/15
29	1/4	3/12	4/15
30	1/4	3/12	4/16

Of course, these strategies should be presented to students one at a time before the game. The game becomes a convenient way to practice different strategies. It also encourages participants to think about fractions rather than follow some well-established algorithm, thus helping to develop numerical fluency.

INVESTIGATING A PATTERN

Analyzing **table 1**, we notice that the fourth guest would get 1 whole bar at each table, the tenth guest would get an amount equal to 1/2 bar at each table, and the sixteenth guest would get an amount equal to 1/3 bar at each table. It would be natural to ask whether the twenty-second guest would get an amount equal to 1/4 bar at each table, the twenty-eighth guest would get an amount equal to 1/5 bar at each table, and so on.

Most teachers at our seminars, when presented with this question, were unsure how to start. This moment of hesitation may evolve into a discussion about problem-solving strategies. The best advice given to me by my professor during the first year at the university was this: "When you do not know what to do, do what you can." What can we do? One possible answer is to collect more data by altering the numbers of bars.

Table 2 lists the numbers when 1 bar is on the first table, 3 are on the second table, and 4 are on the third table. We can see that the sixth guest would have exactly 1 bar at each table, the fourteenth guest would have an amount equal to 1/2 bar at each table, the twenty-second guest would have an amount equal to 1/3 bar at each table, and the thirtieth guest would have an amount equal to 1/4 bar at each table.

Table 3 shows the situation when 2 bars are on the first table, 4 are on the second table, and 5 are on the third table. (Numbers 2, 3, and 5 are suggested to eliminate the suspicion that the observed pattern would work only if there is 1 chocolate bar on one of the tables or if there are consecutive numbers of chocolate bars.) Here the ninth guest would have exactly 1 bar at each table, the twentieth guest would have an amount equal to 1/2 bar at each table, and the thirty-first guest would have an amount equal to 1/3 bar at each table.

Can we find a pattern here? In the first version of the game, there are 6 bars in all. Every sixth guest, beginning with the fourth, is faced with an equal amount of chocolate at each table. In the second version, there are 8 bars in all, and every eighth guest, beginning with the sixth, is faced with an equal amount of chocolate at each table. In the third version, the same is true for every eleventh guest, starting with the ninth.

Notice that in all our examples, there are exactly three tables, and the numbers of bars on each table (1, 2, 3; 1, 3, 4; 2, 4, 5) *have no common factor greater than 1.*

When exploring other such configurations, one may notice that if this game involves n chocolate bars in all, then guests number $(n - 2)$, $(2n - 2)$, $(3n - 2)$, $(4n - 2)$, and so on, will all be faced with an equal amount of chocolate at each table, or a three-way tie.

More formally, it can be stated as the following hypothesis:

> If this game involves n chocolate bars in all, every nth guest starting with guest number $(n - 2)$ is faced with an equal amount of chocolate at each table.

Can we justify this?

Table 3 This grid shows how much chocolate every new guest would have received at each of three tables, one with 2 chocolate bars, one with 4 bars, and one with 5 bars to share. The unshaded cell shows the table that was chosen by this guest.

Number of the Guest	The First Table (2 Chocolate Bars)	The Second Table (4 Chocolate Bars)	The Third Table (5 Chocolate Bars)
1	2	4	5
2	2	4	2 1/2
3	2	2	2 1/2
4	2	2	5/3
5	2	4/3	5/3
6	1	4/3	5/3
7	1	4/3	5/4
8	1	1	5/4
9	1	1	1
10	2/3	1	1
11	2/3	1	5/6
12	2/3	4/5	5/6
13	2/3	4/5	5/7
14	2/3	4/6	5/7
15	2/3	4/6	5/8
16	2/3	4/7	5/8
17	2/4	4/7	5/8
18	2/4	4/7	5/9
19	2/4	4/8	5/9
20	2/4	4/8	5/10

THE "WHYS"

Suppose the game begins with n chocolate bars, distributed among three tables, with at least 1 bar on each table.

1. *Why "Every nth Guest?"*

The number of bars on a particular table becomes the numerator of all the fractions in the corresponding column because that is the number of bars being evenly divided among a growing number of guests. The number of guests becomes the denominator in those fractions. See **table 1**. Only one guest joins the first table between the fourth guest (who would get exactly 1 bar at each table) and the tenth guest (who would get an amount equal to 1/2 bar at each table). At the same time, two guests join the second table and three guests join the third table. In fact, to get to the next "same amount" at each table, we need to add the numerator to the denominator. That is, the total number of the "new" guests will be equal to the total number of chocolates at all three tables.

2. *Why "Starting with Guest Number* (n – 2)*?"*

If the number of guests seated at each table is one fewer than the number of bars on that table (so the number of seated guests is $n - 3$), the next guest (guest $n - 3 + 1 = n - 2$) will be faced with exactly 1 bar at each table.

SUGGESTIONS FOR FURTHER INVESTIGATION

What If the Number of Tables Is Different from Three?

At every table, when the number of guests is one fewer than the number

Table 4 This grid shows how much chocolate every new guest would have received at each of two tables, one with 2 chocolate bars and the other with 3 bars. The unshaded cell shows the table that was chosen by this guest.

Number of the Guest	The First Table (2 Chocolate Bars)	The Second Table (3 Chocolate Bars)
1	2	3
2	2	1 1/2
3	1	1 1/2
4	1	1
5	2/3	1
6	2/3	3/4
7	2/3	3/5
8	2/4	3/5
9	2/4	3/6
10	2/5	3/6
11	2/5	3/7
12	2/5	3/8
13	2/6	3/8
14	2/6	3/9
15		
16		

of bars on that table, there will be exactly 1 bar available for the next guest. We have to add numbers that are one fewer than the numbers of bars on each table. In this way, the next guest will have exactly 1 bar available at each table. Mathematically, "the next guest" can be expressed as "+ 1." Thus, for three tables, the number of the first guest who would have exactly 1 bar can be found as $(a - 1) + (b - 1) + (c - 1) + 1$, where a, b, and c are the number of bars on the first, second, and third tables, respectively.

Table 4 shows how the game would proceed if there were two tables, with 2 and 3 bars, respectively. The fourth guest is the first one who would get 1 bar at either table. After that, every fifth guest will be faced with the same amount of chocolate at each table. This leads to a generalization of our hypothesis:

Assuming that there are t tables with the total of n chocolates on them (each table has at least 1 chocolate bar and there is no common factor greater than 1 for all the numbers of chocolates on the tables), every nth guest starting with guest number $(n - (t - 1) = n - t + 1)$ is faced with an equal amount of chocolate at each table.

What about Common Factors?

Table 3 illustrates the numbers of bars on two out of three tables that have a common factor greater than 1. What if the numbers of bars on *all* the tables have such a factor? We leave it to the reader but provide a heavy hint. Consider, say, 3, 6, and 9 bars on three tables; the common factor is 3. Try the game now. Then (mentally) melt groups of 3 bars into single bars three times as big. You will find that you have just reproduced the first version of the game described here (1, 2, and 3 bars), with these bigger bars as units.

MAKING FRACTIONS PEDAGOGICALLY FLEXIBLE

The MCP can be used in any classroom to review strategies for comparing fractions. It will give students a good opportunity to think about fractions and to make sense of them, rather than escaping to the use of common denominators or to decimal representations. This game will encourage students to talk about fractions.

In a wider sense, this game can also serve as a vehicle for expanding students' powers of reasoning and generalization. This activity is also pedagogically flexible in that a teacher can oversee the game and give less confident students a chance to explain easier cases. Students can be asked to keep a record. Then, after playing a few times with the various numbers of chocolate bars and tables, they can be asked to analyze the numerical patterns, formulate hypotheses, and justify their hypotheses. Playing this game and analyzing the numerical patterns produced are accessible to most middle school students, and the most advanced students will be challenged by constructing precise generalizations and solid explanations for the observed patterns.

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activity sheet

Name ______________________________

HOW MUCH CHOCOLATE WOULD YOU GET?

There are three tables at the Maximum Chocolate Party, and there are 1, 2, and 3 chocolate bars (identical in size and flavor) on those tables. See the diagram below. Each guest, on arrival, chooses a table and sits down. When the last guest is seated, the chocolate from each table is divided equally among those sitting at the table.

When *you* arrive, analyze the tables as if you were to be the last guest, to maximize your chocolate intake.

Complete the table below. Decide how much chocolate you would get at each table, and circle the highest number (if the numbers are equal, choose one as you wish).

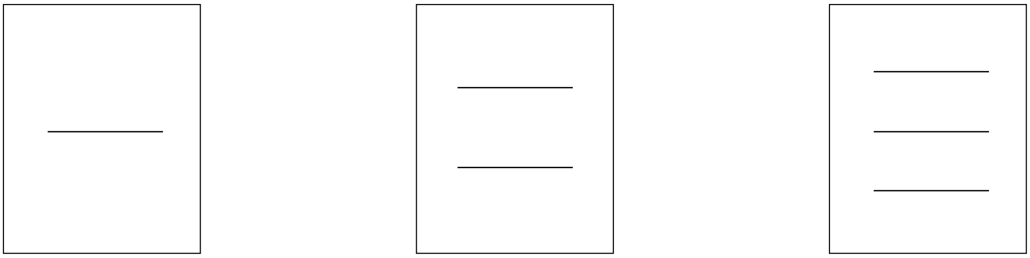

Table 1 Table 2 Table 3

Guest Number	Amount of Chocolate (in Chocolate Bars) the Guest Would Receive at Each Table		
	Table 1	Table 2	Table 3
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			