THE WHOLE STORY: UNDERSTANDING FRACTION COMPUTATION

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Anticipate and address errors that arise when fractions are placed in context and illustrated with models.

What does it look like to *understand* operations with fractions? The Common Core State Standards for Mathematics (CCSSM) uses the term “understand” when describing expectations for students’ knowledge related to each of the fraction operations within grades 4 through 6 (CCSSI 2010). Furthermore, CCSSM elaborates that students should be able to perform fraction computations using visual models and when such problems are presented in context.

These questions then arise: What does it mean to “understand fraction operations”? What types of issues occur that might not be encountered otherwise when solving problems in context and using visual models? What follows are our experiences working with prospective teachers to describe their errors and how those errors might be addressed. Because the prospective teachers were our students, we will refer to them throughout as “students.”

Many examples of student work can be found that demonstrate a misunderstood procedure. Consider one such example in this explanation evaluating 1 3/4 ÷ 1/2:

> I first started to make 1 3/4 not a mixed fraction, so I made it 7/4, which I’m not sure if it’s right. And then to multiply it, I flipped the 2 and the 1. And then I was going to try to cross multiply but I got confused, because it’s not set up as an equation. So


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> then I was trying to figure out how to solve it if they need a common denominator. And I did that and that didn't really look right to me, because I got 7/4 × 8/4, and that didn't seem right. So then I was thinking, well maybe you just multiply the top and the bottom, so I got 14/4. I don't know.

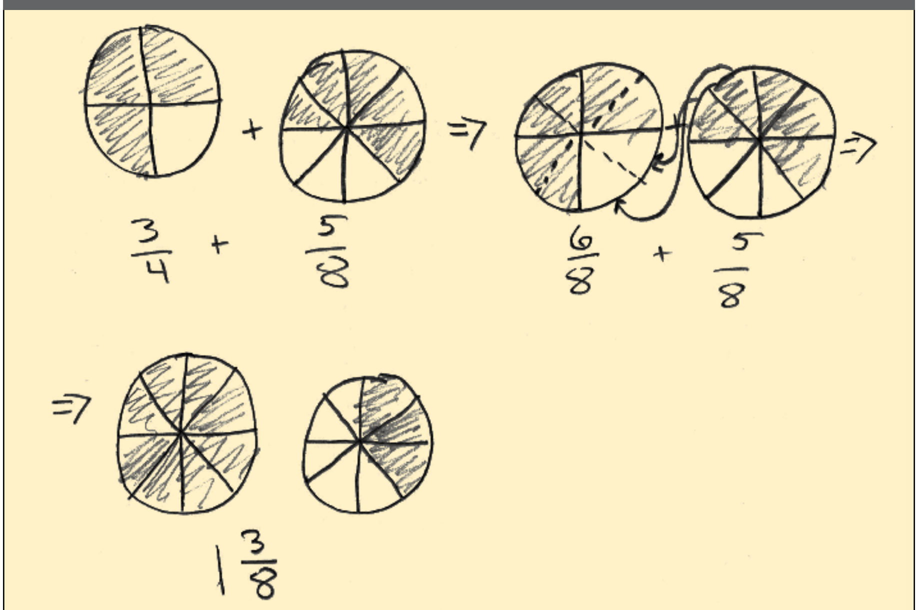
The student attempted to incorporate three different procedures to solve this problem including (1) "flipping" the second fraction and multiplying, (2) finding common denominators, and (3) using cross multiplication. When students know procedures without understanding them, they can easily become confused regarding their use. Also, when an appropriate procedure is applied correctly, students may not know if the solution is correct.

The student quoted above found the correct answer of 14/4, although she was still unsure of her result. Would this student have had more success if the problem had been situated within a context? Our position is that she would. The purpose of this article is to uncover other issues that emerge when understanding fraction operations is sought through presenting problems in context.

Contextualized problems that require fraction operations can help students make sense of the numbers and procedures (Reed 1999; Sharp and Adams 2002). Using word problems enables students "to understand why a procedure works because each step can be related to properties of objects that support the procedures" (Reed 1999, p. 40). With NCTM (2000) and CCSSM (CCSSI 2010) advocating for students to understand fraction operations and for teachers to create word problems for them, it is increasingly important to develop students' understanding of both the operations and the numbers.

According to CCSSM, students in grades 4 through 6 are to base understanding of operations with fractions on operations with whole numbers. Cognitively Guided Instruction (CGI) (Carpenter et al. 1999) defines structures for whole-number operations that can be applied to fractions. Some of those structures are particularly useful in anticipating student errors.

Fig. 1 Students typically use a visual model that supports the context of the problem when first exploring fraction operations based on contexts. For this problem, the context describes eating 3/4 of a pizza, then 5/8 of another pizza.



FRACTION ADDITION

Problems that start with one amount and increase by another amount are called *join problems* (Carpenter et al. 1999). For example, consider the following expression and an associated word problem:

$$\frac{3}{4}+\frac{5}{8}$$

> Gabriella went to a pizza parlor and ate 3/4 of a medium cheese pizza. Then she ate 5/8 of a medium pepperoni pizza. How much pizza did Gabriella eat altogether?

The answer to the total amount of pizza eaten resulted from adding 3/4 and 5/8, or increasing 3/4 by 5/8.

We observed many students' models and found that many typically use a drawing that supports the context of the problem when first exploring fraction operations based on contexts (see **fig. 1**). For example, a circle is likely to be used rather than a number line or a set model to solve a problem involving pizza. When looking at a drawing that illustrates adding 3/4 + 5/8, each fraction is first represented from its own whole. For this problem, the whole for each fraction is 1, or in this case, 1 same-size pizza. In the cheese pizza, each 1/4 piece in the 3/4 is cut in half, so that the eighths are consistent with the pieces in the 5/8, and then they are combined.

Students who solve this problem correctly typically combine the "pizza slices" to make 1 whole pizza with 3/8 of another pizza remaining, or 1 3/8 total pizza. It is important to note that students represent their solution as a mixed number directly rather than writing a fraction greater than 1 as an intermediate step. This is likely the result of the context

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within which the problem was situated. If students do not use context to solve the problem but use straight computation, they typically add the numerators to get a fraction greater than 1. They *then* try to make sense of their answer of 11/8. At this point, language becomes an integral part of making sense of the solution.

Understanding wholes for fractions and defining them are difficult for students. For example, students may know that the answer is 11/8 but not understand the differences in describing this amount as 11/8 pizzas versus 11/8 of the two pizzas (Lamon 1996; Tobias 2013). In this case, the answer would be either "11/8 of *a* pizza" or "1 pizza and 3/8 of *another* pizza." If the student incorrectly gives "11/8 of the *2 pizzas*" as an answer, the student is providing an incorrect answer that describes more than 2 pizzas. This incorrect answer is a result of solving the addition problem in context (Tobias 2009). This error would not be common if the same computation occurred outside of a context.

FRACTION SUBTRACTION

Similar trends are found when subtracting fractions. Two common subtraction forms are (1) *separate problems* and (2) *compare problems* (Carpenter et al. 1999). Separate problems are often described as "take away" subtraction and include decreasing one amount by another amount. Compare problems involve relating two amounts to each other to determine the amount that is greater.

Consider this subtraction problem:

$$\frac{3}{4}-\frac{1}{2}$$

Research supports the efficacy of having students author their own word problems as a way to deepen their understanding of the operations being studied (Alexander and Ambrose 2010; Alibali et al. 2009). We have used this strategy with our students and an interesting common error emerges.

Consider the following separate word problems that students might write to represent the situation.

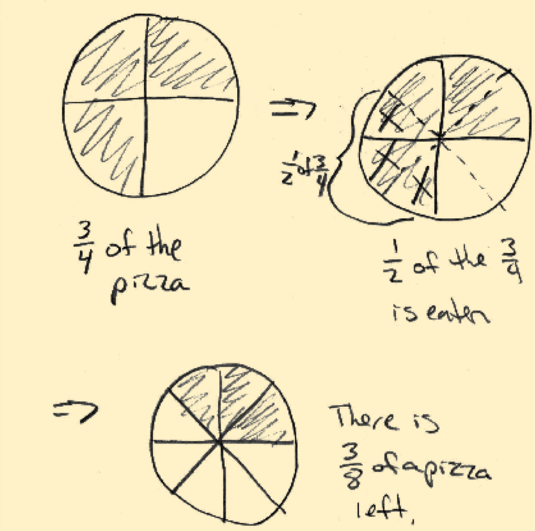
1. Darrell has 3/4 of a cheese pizza leftover from his birthday. He eats 1/2 of the leftover pizza. How much pizza does Darrell have left?
2. Aubrey had 3/4 of a pizza in a box. Before she began to eat, her dog ate 1/2 of a pizza right out of the box. How much pizza did Aubrey have left after her dog ate some?

Both problems describe a situation in which an amount is taken out or used, and both use the fractions 3/4 and 1/2. Employ a visual model to solve each problem before proceeding. Do you get the same answer for both situations?

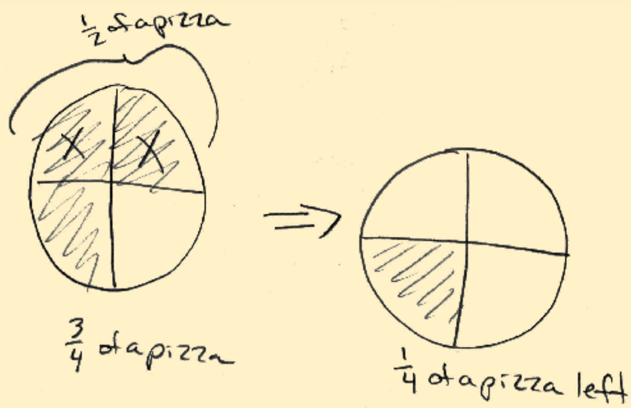
Consider the visual model in **figure 2a**, which is representative of students' work on problem 1. This particular student started by drawing 3/4 of a circle (probably to represent 3/4 of a cheese pizza leftover from Darrell's birthday). Next, the student found 1/2 of the 3/4, which is also shown in the figure. The student then crossed out the part of the pizza representing 1/2 of the 3/4; 3/8 of a pizza was the correct final answer for the first question.

The work to represent problem 2 also started with a circle that was 3/4 shaded (see **fig. 2b**). In this problem, the dog ate 1/2 of an entire pizza from what was left so 1/2 of the circle is marked as eaten. After 1/2 of the circle is crossed out, the remaining 1/4 of the circle, or 1/4 of the pizza, is given as the answer and is correct.

Fig. 2 These two fraction problems, in which the result is smaller than the initial amount, require different representations and calculations.



(a) 1/2 of 3/4 (multiplication)



(b) 3/4 – 1/2 (subtraction)

Each student's solution starts with 3/4 and then an amount is subtracted from that. The first student took away 1/2 of the 3/4 of the pizza, whereas the second student took away 1/2 of an entire pizza. The difference between the methods is what the second fraction in each question refers to.

Which solution makes sense? They both make sense and are correct for their context, but only the second solution matches the expression 3/4 – 1/2. Problem 1 is represented by the calculation

$$\frac{3}{4}-\left(\frac{1}{2}\times\frac{3}{4}\right),$$

not 3/4 – 1/2, as was intended. The reason for this difference is that the second fraction in each question refers to different wholes. This misconception does not emerge when subtracting whole numbers in context or when subtracting fractions devoid of context, even when visual images are used.

The common error of believing that problem 1 can be represented by 3/4 – 1/2 occurs when students are developing an understanding of subtraction as "take away" *in context;* we have found it helpful to anticipate and address this particular error. Interestingly, from our experiences, the error occurs less frequently with word problems that use a clearly defined unit, such as a gallon of tea or a yard of fabric, than with a less clearly defined unit, such as a pizza or a pitcher of tea. The errors are also limited to occurring with separate problems rather than compare problems.

We have observed that using separate problems with a pizza context, for example, highlights the misconception and gives us an opportunity to address and resolve the issue. However, we do so only after considerable discussion, which allows us to work with students to contemplate the idea of the whole in context. We have found that expecting the error and being cognizant of problem types are useful in discussions with students.

Presenting word problems to students and allowing them to create their own strategies using pictures afford an opportunity for conceptual understanding to occur. With addition and subtraction situations, one underlying premise within these two operations is that the amounts represented are from a whole that is the same size. By giving students contextualized situations, they can develop an understanding of this idea because they can discuss actual situations.

For example, students will begin to make sense of their error using a problem with gallons of iced tea and then go back to the problem with pizza to see if they can sort out their difficulty in keeping the *whole* consistent. Without context, students may not necessarily understand that the fractions being added or subtracted and the resulting sum or difference are defined by the same size whole.

FRACTION MULTIPLICATION

For our discussion with multiplication contexts, we will focus on grouping situations (Carpenter et al. 1999). Grouping involves finding the total when the numbers of groups and the size of each group are known. Describing wholes for fraction multiplication problems is different from adding or subtracting. Addends in an addition problem refer to the same size whole, whereas factors in a multiplication problem do not. Consider this word problem representing a grouping situation for

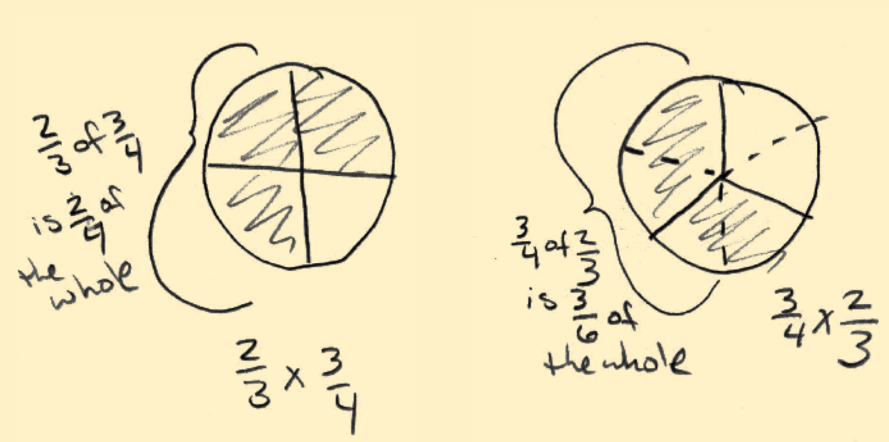
$$\frac{2}{3}\times\frac{3}{4}.$$

> Sue has 3/4 of a pizza leftover. Jim ate 2/3 of the leftover pizza. How much of a whole pizza did Jim eat?

When defining wholes with multiplication, common convention tells us that the multiplication sign can be interpreted as meaning "of." With grouping situations, using whole numbers such as 6×5 refers to 6 groups of 5. Likewise, if we look at our problem and ask what we are finding a part of, we start with 3/4 and take 2/3 of that amount. To represent the word problem with a model, a student draws 3/4 of a whole and then finds 2/3 of the 3/4, as opposed to drawing 2/3 of a whole and finding 3/4 of the 2/3 (see **fig. 3**).

Although the result is the same, the visual model is much different. Our students are able to make sense of this difference when they discuss the contexts of the problems rather than when they rely solely on the visual image. We found that our students can imagine Jim eating 2/3 of Sue's leftover pizza in the problem above more readily than trying to visualize a representation of the problem devoid of context. (Our students have said that they find themselves creating contexts for problems that are presented as straight computations and using pictorial-based strategies to solve them.)

Fig. 3 These visual models show how the order of the factors in fraction multiplication yields different representations, although the result is the same.



(a) 2/3 of 3/4 **(b) 3/4 of 2/3**

We also found that our students seem to develop a deeper understanding of the operation as they discuss situations that are modeled by $2/3 \times 3/4$ versus $3/4 \times 2/3$. This may stem from the fact that when evaluating $3/4 \times 2/3$, one needs to cut the 2/3 to take 3/4 of that amount. Whereas with $2/3 \times 3/4$, the pieces are already situated so that it is easy to take 2/3 of 3/4 directly without dividing up the 3/4 further into more and smaller pieces.

Finally, the problem specifies that we are looking for the amount of a whole pizza that Jim ate. (See **fig. 3a**.) He ate 2/3 of 3/4; if we compare Jim's amount with the whole pizza, then he ate 1/2 of a pizza because 2 of the 3 fourth-size pieces represent 2/4, or 1/2, of the whole.

We started with 3/4, so this representation was in terms of the original whole, which was 1 pizza. Taking 2/3 of that amount meant that we had to find 2/3 of 3/4, making 3/4 the new whole that was now to be used. Finally, to answer the question, the whole changed back to the original whole of pizza, so the answer was 1/2 of a pizza. Similar characteristics can be seen with fraction division situations.

FRACTION DIVISION

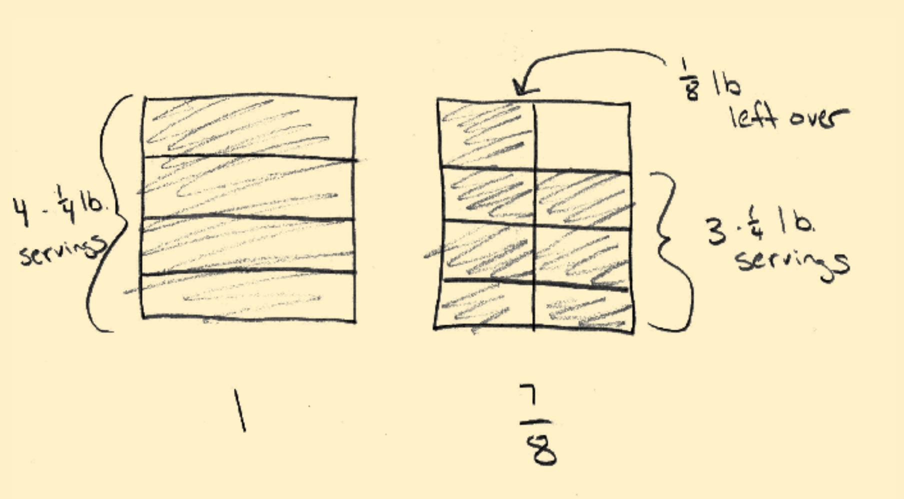
Two common problem types for division situations are *partitive* and *measurement* meanings (Carpenter et al. 1999). With partitive division, we know the total and the number of groups. We seek to determine the amount in each group. With measurement division, we know the total and the amount in each group and seek to determine the number of groups that can be made. It is most common to use contexts supporting measurement division when dividing fractions by fractions because sharing among a part of a group is difficult to visualize (Dixon et al. 2011). Therefore, we will focus on measurement division exclusively.

Consider this measurement division word problem for the following fraction:

$$1\frac{7}{8}\div\frac{1}{4}$$

> There are 1 7/8 pounds of fudge at the candy store. How many 1/4 pound packages can be made from this fudge? What part of another package will be leftover?

Fig. 4 This visual model, determining how many 1/4 pound servings are possible from 1 7/8 pounds of chocolate, must consider that the 1/8 pound of leftover fudge is 1/2 of a 1/4 pound serving.



When solving this problem, we start with the original amount of 1 7/8 pounds and see how many 1/4 pound packages can be made from the 1 7/8 pounds of fudge (see **fig. 4**). In other words, we are looking to see how many groups of 1/4 pound each can fit into the total of 1 7/8. The solution is the number of groups.

Start with 1 7/8 pounds and section off groups that are 1/4 pound. This illustrates that both fractions are out of the same whole pound. From this, there are seven servings of 1/4 pound each that can be made, with some leftover. This leftover amount is often difficult to describe. Many students will say this fraction represents 1/8 because that amount is 1/8 pound. Students need to understand that this question is asking what part of a 1/4 pound package remains, and the remainder needs to be described based on the package size. The package size becomes the new whole; therefore, the 1/8 pound leftover needs to be recognized as 1/2 of the 1/4 pound package. Thus, the correct answer to this problem is 7 1/2, not 7 1/8. The context of the problem leads to the correct interpretation of the whole.

In early work with division using whole numbers, students are asked to find a remainder in terms of a whole number. For example, if solving 9 ÷ 4, the solution of 2 remainder 1 is written as 2 R1. When students move to fraction situations, they must coordinate what part of another group remains, rather than just describing what remains in terms of the original whole. Instead of writing the solution as 2 R1, this answer becomes 2 1/4 because students are looking for groups of 4; in other words, 4 becomes the new whole.

When students move to fraction division, the remainder is found in a similar way by referring to how much of another group they are seeking. In the solution of 7 1/2 above, the 1/2 is how much of a 1/4 pound package can be made from the leftover fudge, or

$$\frac{\left(\frac{1}{8}\right)}{\left(\frac{1}{4}\right)}.$$

Our students rely on the context of the problem even more than the visual image to convince themselves that the answer is 7 1/2 rather than 7 1/8 as they describe how much of another package they can make.

MITIGATING ERRORS

Working with word problems provides an opportunity for students to develop an understanding of what each number represents. By being exposed to various contexts for operations, our students used their previous knowledge of whole numbers to make sense of working with these same types of situations with fractions. Using procedures alone in computing with decontextualized problems results in a well-documented set of errors as illustrated through the example of the thinking of the student shared at the start of this article. On the other hand, using visual models to solve problems provided in context can result in another set of errors related to incorrectly defining the whole. We have found that the key is to anticipate these errors and use them as springboards to learning. This allowed our students to make connections in mathematics and develop an understanding of what it means to operate with fractions.

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