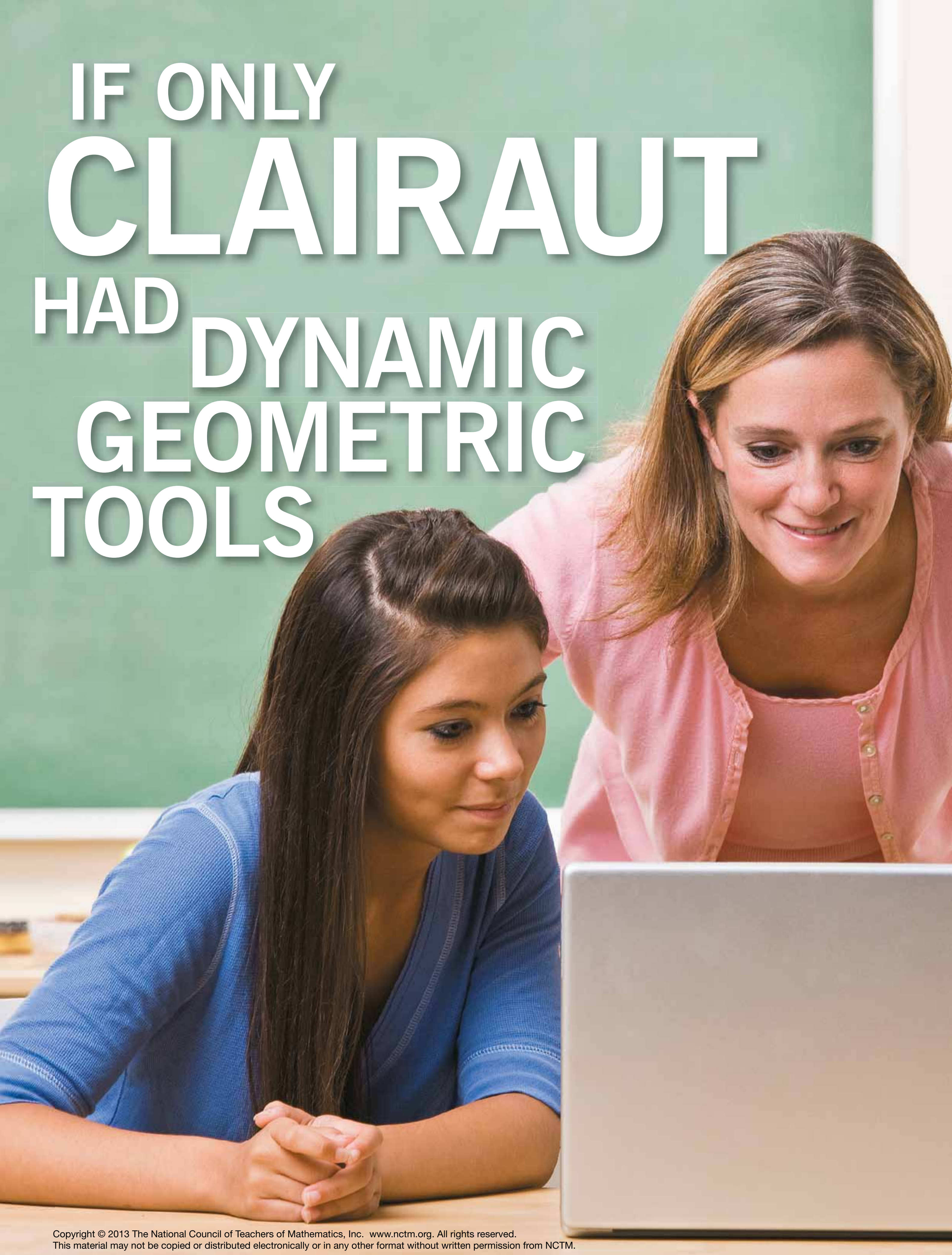
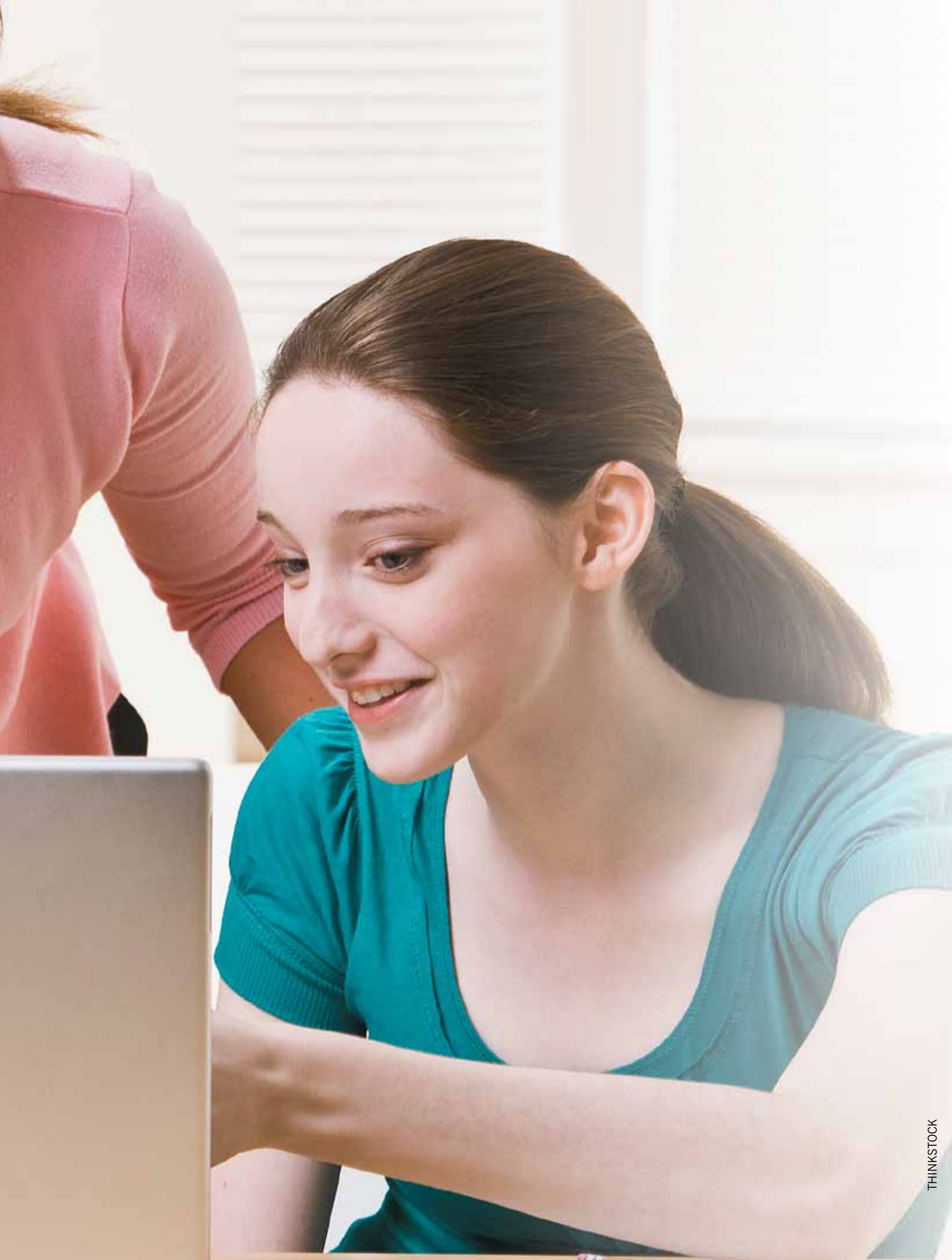
IF ONLY CLAIRAUT HAD DYNAMIC GEOMETRIC TOOLS



Using Clairaut's historic-genetic approach and dynamic geometry tools in middle school can develop students' conceptual understanding before they encounter formal proof in geometry.

Hyewon Chang and Barbara J. Reys



THINKSTOCK

Geometry is a major area of study in middle school mathematics, although it typically receives far less attention than topics such as algebra and rational numbers. In fact, middle school and secondary students have difficulty learning important geometric concepts although these concepts are a much more visible part of their daily lives than algebra. Some conjecture that students experience difficulty learning school geometry and connecting it to everyday geometry because of the premature focus on rigorous and formal proof before meaningful conceptualization of key ideas (Battista 2009).

In the geometry strand of the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010), middle school students are expected to "use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles" (p. 56) In grade 8, students are expected to "Explain a proof of the Pythagorean Theorem and its converse" (p. 56). These statements reflect the need to develop a conceptual foundation based on exploration, discovery, and explanation *prior to* an emphasis on formal proof. The activity described below supports these goals.

We consider two opportunities for increasing the conceptual focus on geometry in the middle grades. One is to rethink how to introduce and develop concepts of school geometry. In particular, we consider Clairaut's approach that emphasizes engaging student curiosity about key ideas and theorems instead of directly teaching theorems before their application in real life. The second related opportunity is use of computer technology that enables and promotes student exploration and engagement.

According to Clairaut's method, teachers can support student engagement by asking them to consider ideas and investigate their truth.

CLAIRAUT'S APPROACH

Clairaut, a famous eighteenth-century French mathematician and astronomer, prepared a geometry text that drew on what he called the natural human thought process. His approach is characterized by combining logical and intuitive elements, instead of building elements of geometry on a purely logical basis. In fact, Clairaut criticized Euclid's logico-deductive approach, which he felt made learning difficult because of its dull and boring development.

Clairaut was also critical of an alternative that addressed the limitation of Euclid's approach by including applications following formal work on theorems. Instead, Clairaut drew on the historic-genetic principle, which dated from the seventeenth century and recognized that discoverers themselves were beginners in the history of mathematics. He theorized that the discoverers' trajectory of development—initiated by a particular need and formalized gradually—might also be beneficial to later generations of beginning learners. The result was his masterpiece *Elémens de Géométrie,* published in 1741.

Clairaut's approach to geometry instruction can be characterized by two principles: practical necessity and intuition-driven. It is in contrast to Euclid's rigorous approach, which moves toward formalization before learners are necessarily motivated by a need to know. Euclid's method starts with definitions, axioms, and postulates that often do not excite or engage the average learner. On the other hand, Clairaut begins with practical necessity, mimicking the motivation of the inventors of geometry, as the first step in exploring an idea. Clairaut's approach does not insist on rigorous exactness of mathematical proof. Rather, justification is intuitive and used only when necessary.

Although Clairaut acknowledged criticism that his approach relied too much on the testimony of the eyes and neglected the strictness of demonstration, he defended the approach. He argued that the focus for beginners should be on big ideas, not propositions whose truth may be discovered by the smallest degree of attention. For example, consider these statements:

- "In any triangle, the greater side subtends the great angle."
- "A circle does not cut another circle at more than two points."
- "If two circles touch one another, then they will not have the same center."

Although Euclid labored to demonstrate these ideas, Clairaut thought trivial propositions such as these did not warrant students' attention by a strict demonstration. Stamper (1909) noted that Clairaut "was ready to sacrifice logic for the sake of interest and practical necessity."

Clairaut's ideas are particularly appropriate for middle school mathematics where formalization is not the goal. As an example of Clairaut's theory, we consider the idea that the sum of the interior angles of a triangle is constant. Unfortunately, most students are not curious about this idea and are not given the opportunity to discover it. According to Clairaut's method, teachers can support student engagement by asking them to consider ideas and to investigate their truth. For example, the teacher might pose the following challenge to students:

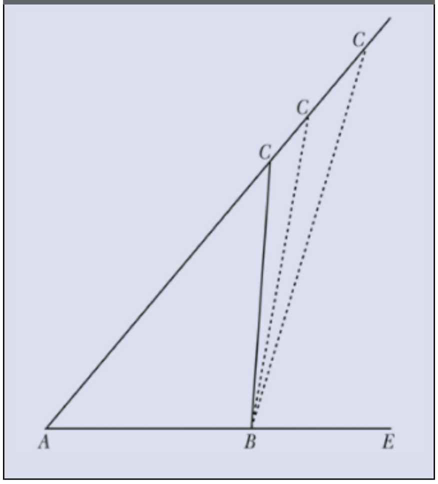
> It has been said that the sum of the angles of a triangle (any triangle) is equal to the sum of the angles of any other triangle. Is this true?

Because we cannot be satisfied with measured values of the angles of a triangle (measurement is never exact), we must look for another way to investigate the claim. Clairaut's strategy can be suggested as a method. He used the reasoning that two of three angles in a triangle decide the size of the remaining angle. That is:

> Using **figure 1**, suppose vertex *C* goes up along *AC*. It is certain that angle *C* closes gradually and angle *B* opens gradually the other way. We can assume the portion of decrease of angle *C* and increase of angle *B* are the same, so whatever the slope of *BC* to *AC* and *AB*, the sum of the three angles of triangle *ABC* is always constant. (Clairaut 1741)

The assumption about the equivalence of the decrease in angle *C* and

Fig. 1 The idea that the sum of interior angles of a triangle is constant can be produced by changing the slope of *BC* to *AC* and *AB*.



complimentary increase in angle *B* is natural for students. At a later time (e.g., high school), formal proof of the theorem can be developed using alternate angles in parallel lines. Although this exploration might be of interest to some students, a current tool such as dynamic software can provide a more motivating, interactive, and student-driven method for students' exploration of claims.

DYNAMIC GEOMETRY TOOLS AND CLAIRAUT'S METHOD

In Clairaut's time, the implementation of his approach was constrained by the tools that were available during his lifetime. In the triangle example in **figure 1**, for instance, the use of dynamic geometry software allows the quick generation and animation of examples. Today, we can use his theory via a dynamic environment aided by technology. In other words, thanks to the instruments of technology, his method can be reborn. In this section, we illustrate a few examples of topics to which Clairaut's approach using dynamic geometry tools are appropriate (e.g., properties of isosceles triangles, the sum of angles at a point on a line, the sum of interior angles of a triangle, and the properties of inscribed angles).

Dynamic geometry software can help students visualize that a particular theorem holds under varying factors (size of angles, length of segment, and so on). Students can change the factors as many times as they want instead of viewing only one static image in the paper-and-pencil environment. In this case, the medium creates an environment where it is easier to recognize, explain, and even generalize geometrical properties.

Examples of Clairaut's approach using a dynamic geometry tool (e.g., The Geometer's Sketchpad®, GeoGebra, NCTM's Core Math

Stamper (1909) noted that Clairaut "was ready to sacrifice logic for the sake of interest and practical necessity."

Tools) are described here. Although the static constraint of print inhibits the full presentation of the examples, we explain the advantage and efficacy in each case and encourage readers to test the ideas in a computer environment. The geometric constructions suggested can be demonstrated by the teacher or developed by motivated students. In either case, the focus is on students learning geometric concepts and properties by manipulating visual shapes and exploring geometrical variants and invariants dynamically.

Example 1

> The sum of angles having a common point on a straight line is 180 degrees. Furthermore, the sum of angles around a point equals 360 degrees.
>
> (Clairaut 1741, LVII and LVIII of the first part)

The exploration in **figure 2**, in which all five angles can be variants, allows students to "see" a geometric property: The sum of the angles around a point is 360 degrees. Students can change the size of any particular angle by pivoting *AB*, *AC*, *AD*, *AE,* or *AF* (without overlapping an adjacent line), yet the sum of the angles is constant. Students conclude that "the sum of angles around a point equals 360 degrees." The same method of exploration is appropriate for angles on a straight line and their sum of 180 degrees.

Example 2

> The sum of interior angles of a triangle is constant.
>
> (Clairaut 1741, LXII of the first part)

The constancy of the sum of three angles of a triangle is quite intuitive (see **fig. 1**), but very surprising. In this example, students see that holding one angle constant while changing another angle affects the third angle. In a dynamic environment, they can see the constant sum of 180 degrees, whatever the shape of the triangle (see **fig. 3**). It is helpful for students to experience and explore this idea by dragging vertices of a triangle before entering into a formal proof of the theorem. The combination of Clairaut's idea of exploration about the constant sum and its realization via a dynamic representation is convincing to students.

Example 3

> Any point on a semicircle forms a right angle when connected to the ends of the diameter of the semicircle. In other words, every

Fig. 2 Students can use software to explore the fact that the sum of angles around a point equals 360 degrees by dragging points to change the measure of the angles. They will notice that their sum is always 360 degrees.

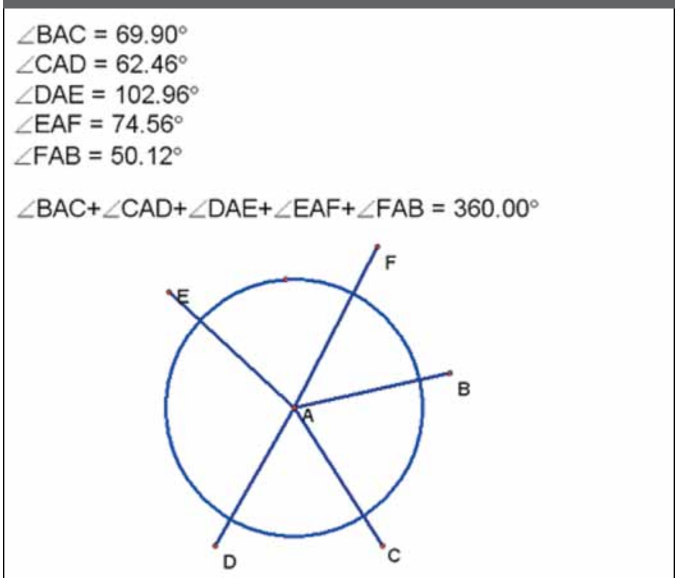
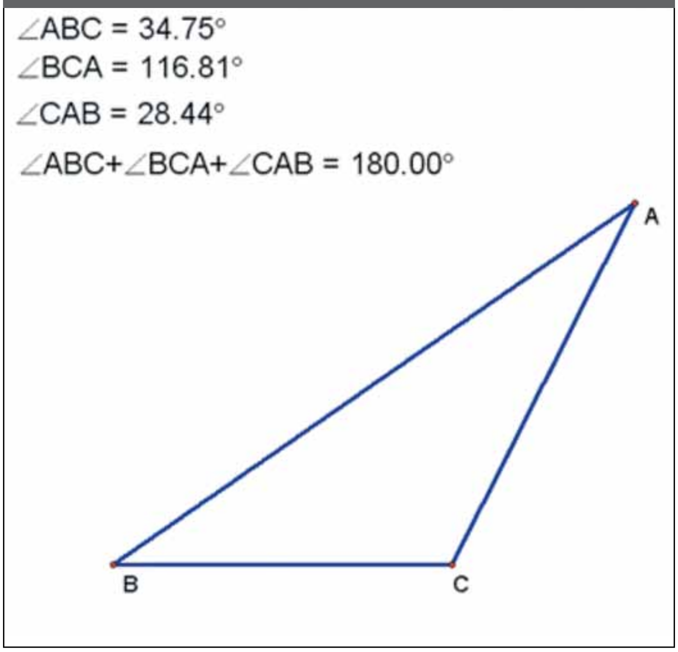


Fig. 3 Exploring triangles dynamically will allow students to conclude that the sum of interior angles of a triangle is constant.



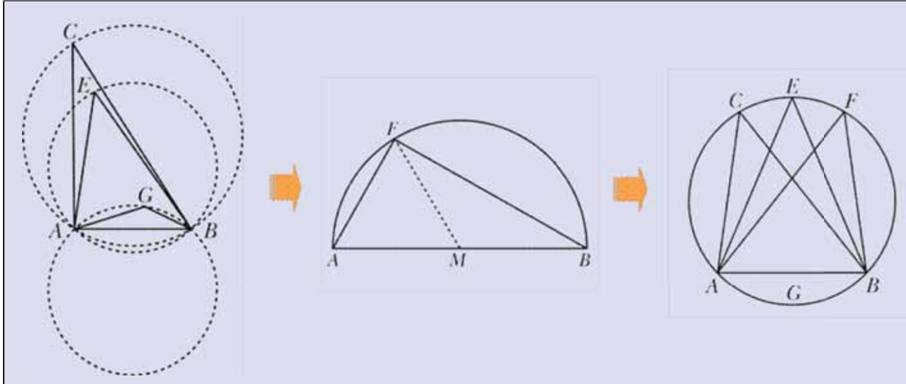
inscribed angle of a semicircle measures 90 degrees.

(Clairaut 1741, XIII of the third part)

Following the steps that Clairaut suggested, one can construct a circumcircle of any triangle with constant *AB*, as shown in **figure 4a**. When the third vertex of the triangle varies, the circumcenters of the triangles *ABC* and *ABE* approach the segment *AB*. The circumcenter of the triangle *ABG*, an obtuse triangle, goes below *AB*. When is the center located on *AB*? What kind of triangle is this?

To answer the question, see **figure 4b**. When circumcenter *M* is located on *AB*, *AB* is a diameter of circle *M* because point *M* is the center of a circle. In other words, sector *AFB* is a semicircle, and segments *MA*, *MF*, and *MB* are all the same length.

Fig. 4 Clairaut suggested these diagrams be placed in sequence to lead explorations of properties about an inscribed angle.



Triangles *AMF* and *FMB* are isosceles and angles *MAF* and *MFA* and *MFB* and *MBF* are congruent, respectively. Therefore, angle *AFB* is a right angle in triangle *ABF*. When using a dynamic geometry tool as shown in **figure 5**, students can confirm that angle *ABC* consistently measures 90 degrees after dragging point *B* along arc *ABC*.

This property allows users to wonder whether the same kind of property might hold for any segment *AB* even though it is not a diameter (see **fig. 4c**).

Fig. 5 The fact that every inscribed angle of a semicircle measures 90 degrees is more convincing to students when they can move point *B* and explore the change, or lack of change.

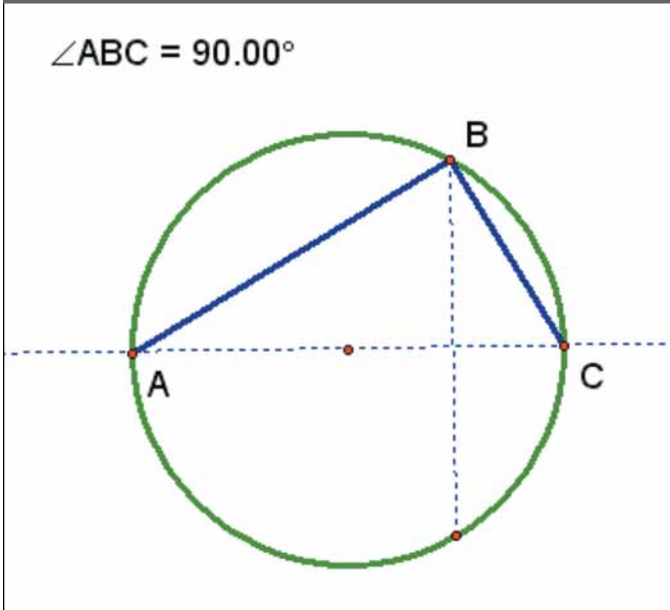
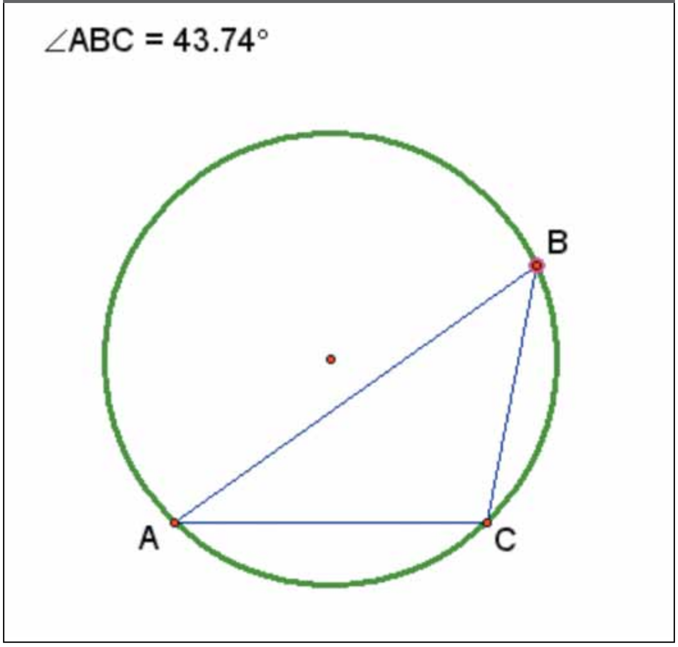


Fig. 6 This sketch helps enforce the idea that all inscribed angles that subtend the same arc are equal.



What if *AB* is not a diameter, but the third vertex varies along its same circumcircle? The inscribed angles *ACB*, *AEB*, and *AFB* will not measure 90 degrees, but will its measure change? Clairaut described this idea as shown in example 4.

Example 4

> All inscribed angles that subtend the same arc are equal.
>
> (Clairaut 1741, XIV of the third part)

Students can explore this idea in a dynamic environment (see **fig. 6**). If segment *AC* is considered fixed, students see and recognize that angle *ABC* is constant after dragging point *B* along arc *ABC*, thereby visually confirming the property.

CLAIRAUT, THE TEACHER, AND TECHNOLOGY

We describe Clairaut's method of capitalizing on students' intuitive ideas about geometry using a dynamic computerized environment. Until recently, Clairaut's ideas could not be fully realized in classrooms. The availability of dynamic geometry software now makes it possible. By pursuing his ideas with current technology, we can engage students in exploration and discovery.

Of course, the teacher's role is crucial to the success of the method. Teachers establish the classroom environment that allows student exploration to flourish. In addition, it is important to identify and create the scenarios under which students can and will explore important ideas. Also crucial is considering when technological tools are more efficient than paper-and-pencil exploration and for which ideas. Teachers orchestrate and monitor the flow of student ideas, prompting students to, for example, recognize invariants. They must also make time for a full discussion and reflection of the ideas following the software-aided exploration.

One key principle in using dynamic software is to identify geometric ideas that can be understood better in the condition of dynamicity. That is, dynamicity helps learners discriminate between variants and invariants. When students change the shape of a geometrical figure by dragging a point, they are expected to notice variants and invariants. In our examples, invariants show the intended property. Therefore, teachers must be ready to draw attention to the invariants and ask, "What changes when you drag a vertex?" or "What appears to stay the same?"

Although dynamic explorations such as those described here are likely to be interesting to students because of their novelty and because they are student-directed, the intended learning outcomes do not happen automatically. They require guided reflection in which the teacher encourages students to think back on the meaning of their own work. Dynamic geometry software is a tool, and the purpose of using it is to learn the concepts and properties of geometry.

Clairaut's approach does not replace the need for formal proof in geometry. In fact, Clairaut provided the logical proofs after introducing intuitive approaches in most cases. At the middle school level, we believe the intuitive approach should be emphasized because it builds student curiosity and can lay the foundation for later formalization. Current technology gives us a perfect opportunity to revisit Clairaut's approaches, thus improving opportunities for student learning in the middle-grades mathematics classroom.

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Hyewon Chang, hwchang@snue.ac.kr, is an associate professor at Seoul National University of Education in South Korea. She participated in developing current national curriculum for mathematics and is interested in research on learning geometry and using history of mathematics for mathematics education. **Barbara J. Reys**, reysb@missouri.edu, is Curator's Professor and the Lois Knowles Faculty Fellow at the University of Missouri–Columbia. She directs the Center for the Study of Mathematics Curriculum, focusing on improving mathematics education through research and development efforts related to mathematics curriculum.