The Distributive Property in Grade 3?

Are third graders ready to connect procedures to concepts of area conservation, distribution, and geometric interpretation?

By Christine C. Benson, Jennifer J. Wall, and Cheryl Malm

The Common Core State Standards for Mathematics (CCSSM) call for an in-depth, integrated look at elementary school mathematical concepts. Some topics have been realigned to support an integration of topics leading to conceptual understanding. For example, the third-grade standards call for relating the concept of area (geometry) to multiplication and addition (arithmetic). The third-grade standards also suggest that students use the commutative, associative, and distributive properties of multiplication (CCSSI 2010).

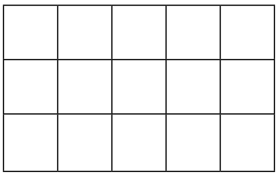
Traditionally, multiplication has been a major topic for third grade. Linked to repeated addition of equal-size groups, multiplication logically follows the study of addition. Introducing rectangular arrays to represent groups (rows) of equal size illustrates both numeric and geometric interpretations of multiplication and naturally introduces the concept of area (CCSSI 2010). Rotating the rectangular arrays illustrates the commutative property of multiplication (see **fig. 1**), and determining the number of tiles needed to build a multicolor rectangle allows students to demonstrate their understanding of conservation of area and to discover a geometric interpretation of the distributive property (see **fig. 2**).

Although multiplication is typically a focus of third-grade mathematics (NCTM 2006), third-grade textbooks usually include few, if any, concepts of area or distribution and no geometric interpretation of the distributive property, which raises at least two questions: (1) Are third graders ready for the reasoning needed to understand these concepts? And, if they are, (2) how can integrated exploration of these topics help students make connections that deepen their conceptual understanding of these topics and others already in the curriculum?

FIGURE 1

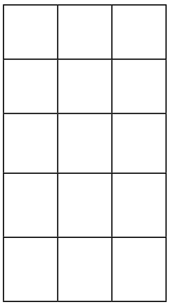
Rectangular arrays encourage the discovery of the commutative property of multiplication.

Three rows of tiles with five tiles in each row



$3 \times 5 = 15$

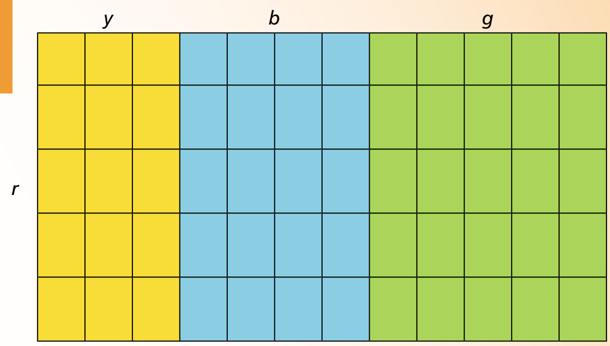
Five rows of tiles with three tiles in each row



$5 \times 3 = 15$

FIGURE 2

A multicolored array allows for an exploration of the distributive property.



$r(y + b + g) = (r \times y) + (r \times b) + (r \times g)$

The problem

To answer the first question and evaluate students' understanding of multiplication and area, we gave an open-ended problem (see **fig. 3**) to a group of beginning third graders. The students had worked with decomposition strategies along with the associative and commutative properties for addition; for example, thinking of 12 + 15 as 10 + 2 + 10 + 5 and seeing 2 tens for 20 and 2 + 5 for 7, to get 27. They also had used skip counting to add equal groups. Their teacher indicated, however, that they had not been introduced to multiplication, conceptually or symbolically. To determine if their natural reasoning would support a more formal exploration of area and distribution, we wanted to observe how students at this level of understanding would solve the given problem.

We gave students colored tiles, graph paper, and colored pencils to help them explore the problem. Their solutions illustrated several different approaches. Some students simply wrote the number sentence 12 + 12 + 12 + 12 + 12 = 60. When asked, they explained that twelve tiles were in each row, and there were five rows, so they added twelve five times. These students illustrated the ability to

FIGURE 3

The problem that beginning third graders receive assesses their natural reasoning.

How many cards are needed?

A club wants to create a card section at the next football game by placing a card on each seat in a certain section. They are planning to use a section of seats that has 5 rows, and they plan to put 3 yellow cards, then 4 blue cards, and then 5 green cards down each row. How many cards altogether will be needed so that each seat in this section has a card on it? Write number sentences to show how you found your answer. Then write a sentence or two explaining your thinking.

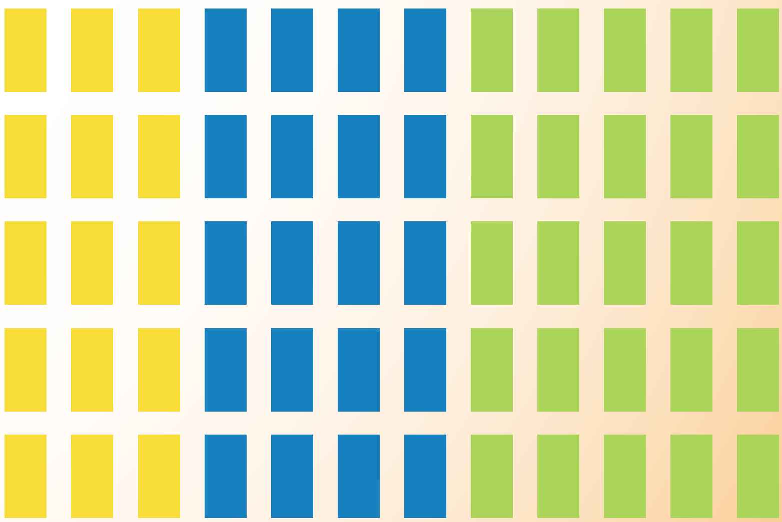


FIGURE 4

Student strategies could be used to develop the distributive property.

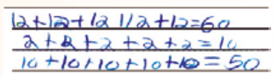
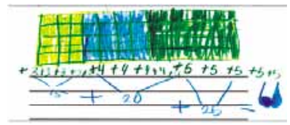


FIGURE 5

Students used repeated addition for each of the smaller rectangles.



visualize equal-size groups; they would be ready to link this to multiplication both conceptually and symbolically.

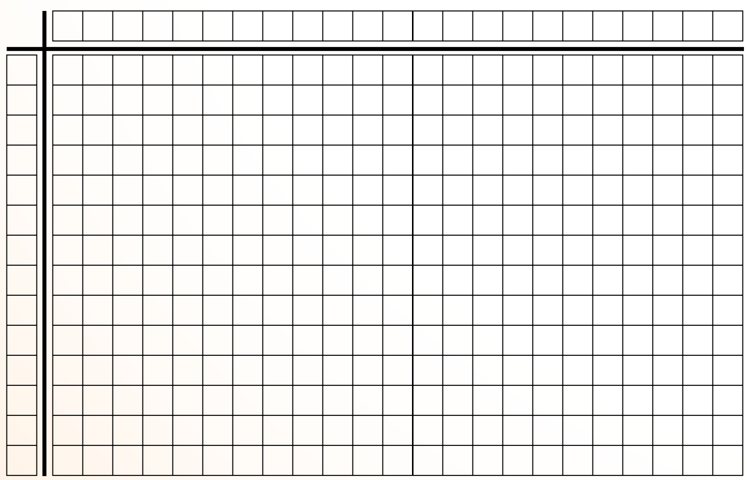
Other students, when asked how they determined the sum was equal to sixty, demonstrated a general understanding of the distributive property. These students explained that within the 5 twelves are 5 twos that would equal ten and 5 tens that would equal fifty; then 50 + 10 = 60 (see **fig. 4**). This decomposition naturally leads to symbolizing the problem as $5(2 + 10) = 5(2) + 5(10) = 10 + 50 = 60$.

Another group of students exhibited a geometric interpretation of distribution. They explained by using color to separate the larger rectangle into smaller rectangles. These students determined how many tiles were in each row of the smaller rectangles, used repeated addition to find the total number of tiles in each smaller rectangle, and then added those totals (see **fig. 5**).

A fourth group of students used the smaller, single-color rectangles to count the number of columns in each of the smaller rectangles so they could skip-count by fives. Although students could have used skip counting by fives in the large rectangle to find the total number of tiles (e.g., 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 = 60), most students using this method indicated that they visualized the problem as three smaller rectangles. They explained that there were three groups of yellow tiles (or 15), four groups of blue tiles (or 20), and five groups of green tiles (or 25). All these students did not necessarily add 15 + 20 + 25 to get the total number of tiles, reverting instead to counting all the tiles to answer the question; but they did exhibit a basic understanding of

FIGURE 6

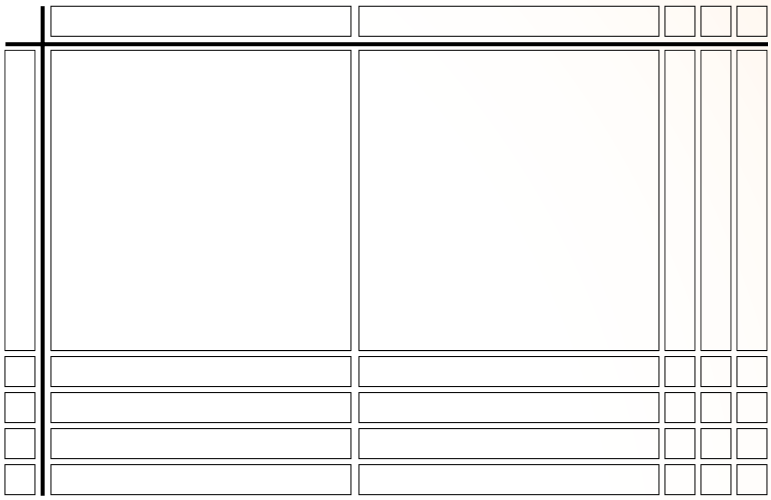
Area models of multiplication allowed students to see equal-size groups.



$23 \times 14 = 322$

FIGURE 7

The area model with base blocks enabled students to make connections to the area model with unit tiles.



$23 \times 14 = 322$

conservation of area. Students' exhibition of intuitive understanding supports their readiness for an integrated exploration of these topics.

Nearly all students demonstrated an understanding of conservation of area, that is, that the area of the large rectangle is equal to the sum of the areas of each of the smaller rectangles. From this, many students understood that five rows of twelve tiles (5×12) is the same as five rows of three tiles plus five rows of four tiles plus five rows of five tiles, or $5(3 + 4 + 5)$, which illustrates the distributive property. When students can add, skip-count, and conserve area, a concrete understanding of the distributive property is intuitive:

> The precursors of…[the] distributive propert[y]…are already there as natural logic, the child's natural habits of mind and the building blocks of higher mathematics. (Goldenberg, Mark, and Cuoco 2010, p. 555)

Finding evidence that such understanding was in place, we proceeded to the second question.

FIGURE 8

The area box model without base blocks provided a transition from the concrete representation of the distributive property with the base blocks to a more abstract representation in this model.

	20 +	3
10	200	30
+		
4	80	12

$23 \times 14 = 322$

The concepts

To answer the question of how to integrate the exploration of these topics, consider the following examples, sequenced to illustrate the connections among multiplication, area, the distributive property, and algorithms currently in most textbooks. When doing these activities, be sure that students

FIGURE 9

The expanded notation method allowed students to see the distributive property numerically.

$$(20 + 3) \times (10 + 4) =$$

$$(20 \times 10) + (3 \times 10) + (20 \times 4) + (3 \times 4) =$$

$$200 + 30 + 80 + 12 =$$

$$322$$

FIGURE 10

The expanded notation vertically enabled students to transition to the standard algorithm while still connecting the concepts learned in the previous models.

$$\begin{array}{rl} 20 + 3 & \\ \times\, 10 + 4 & \\ \hline 12 & (\ 4 \times \ 3) \\ 80 & (\ 4 \times 20) \\ 30 & (10 \times \ 3) \\ 200 & (10 \times 20) \\ \hline 322 & \end{array}$$

FIGURE 11

The partial-products method provided the link between the expanded notation methods and the standard algorithm.

$$\begin{array}{rl} 23 & \\ \times\, 14 & \\ \hline 12 & (\ 4 \times 13) \\ 80 & (\ 4 \times 20) \\ 30 & (10 \times \ 3) \\ 200 & (10 \times 20) \\ \hline 322 & \end{array}$$

are active participants at each step, not just observers (NRC 2001).

Area model with unit tiles

As discussed earlier, building rectangles to illustrate equal-size groups (i.e., rows, each with the same number of tiles) shows multiplication as the area of a rectangle. Here, the area of the rectangle, or the number of square tiles it takes to cover that rectangle, is the product of its dimensions. This is an effective model for single-digit factors, but it is cumbersome with large numbers (see **fig. 6**). So, to be more efficient and reinforce place value, we switch to base-ten blocks.

Area model with base blocks

We still make a rectangle using the factors as dimensions, and the area of the rectangle is still the product. But we use the blocks to group the tens and use the physical dimensions of the factors represented by the blocks to confirm the dimensions of the blocks that represent the product. Geometrically, this is the distributive property. For example, 4 × 12 is illustrated as four groups of ten and four groups of two, showing $4 \times 12 = 4(10 + 2) = 4(10) + 4(2) = 40 + 8 = 48$.

For some factors, like 23 × 14, students may need to regroup to get the final answer (see **fig. 7**). Working with physical blocks before working with pictures allows students to manipulate the blocks, exchanging ten units for a rod or ten rods for a flat. However, students soon just draw pictures, especially on homework. But like before, as the values of the numbers increase, the physical objects and drawings become cumbersome, and we look for something more efficient.

Area box model without base blocks

At this stage, we no longer use base blocks but still use a rectangular box model, which we no longer draw to scale (see **fig. 8**). We split the dimensions of the rectangle into tens and ones, and it is important to make sure students understand how the areas of the sides of this new rectangular prism relate back to the base-blocks rectangles. Students should do several examples both ways, side by side, so they see this important connection of something concrete to something more abstract (Reys et al. 2009, p. 188).

Transition to expanded notation

Once students are comfortable using the box method, we begin a transition to something more abstract by adding expanded notation to the box method. Again, students do both methods side by side for a while to make sure they see the connections (see **fig. 9**). Using the box methods and expanded notation, we can illustrate the distributive property both geometrically and numerically. Eventually, students no longer need to draw the pictures, although some may continue to visualize the box in their minds.

Expanded notation—vertically

Now that students understand what multiplication is and why it is important that we consider place value when multiplying, we begin to transition to the standard American algorithm (an efficient way of multiplying). We continue to use expanded notation, but we rearrange how we write it, so that it starts to look like the standard algorithm (see **fig. 10**). The partial products that are listed match the representations seen in the base-block models, and the connecting line segments help students see the relationship to the arcs in **figure 9**.

Partial products

This is the same thought process we used previously, but we do not write the factors in expanded notation (see **fig. 11**). However, we still consider a digit in the tens place to represent that many tens (20 for example, not just 2).

The standard American algorithm

Continuing to use the distributive property, we multiply the units digit in the second factor by both digits in the first factor, all in the same step (rather than writing it in two steps) (see **fig. 12**). Again, doing both methods side by side for a few problems is important so that students make the connection and understand that the steps in the algorithm are based on using the distributive property.

Students will no longer see the standard algorithm as a meaningless list of steps that must be completed in a certain order.

FIGURE 12

The standard algorithm is an efficient means of multiplying multidigit numbers.

$$\begin{array}{r l} 23 & \\ \times\ 14 & \\ \hline 92 & \text{(Notice that this is 12 + 80)} \\ 230 & \text{(Notice that this is 200 + 30)} \\ \hline 322 & \end{array}$$

Instead, they will understand how it connects to the basic multiplication of single-digit numbers in the area model and to the distributive property.

Extensions

Older students (or young students who need a greater challenge) could use the same models but could change the base. Instead of grouping by tens, they could group by nines, for example. Consider these problems (see **fig. 13**):

$12_{nine} \times 4_{nine}$ and $23_{nine} \times 14_{nine}$

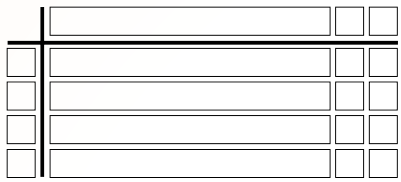
When students make (or draw) the rectangles, they see that the pictures look the same as they did when using base-ten blocks (see **fig. 7**), but the rods are shorter; and to get the final answer, they group by nines instead of tens. When we have students multiply 12 × 4 and 23 × 14 in bases 5, 7, and 8, they see that the rectangular pictures look the same every time. Only the size of the grouping changes when simplifying to get the final answer. This means, and we have students verify, that all the other methods follow as well, as long as we group according to the base we are in. What students can learn later depends on what they have learned before (CCSSI 2010, p. 5), so teachers of older students can use this pattern to help students make sense of algebra as generalized arithmetic.

Area model with algebra tiles

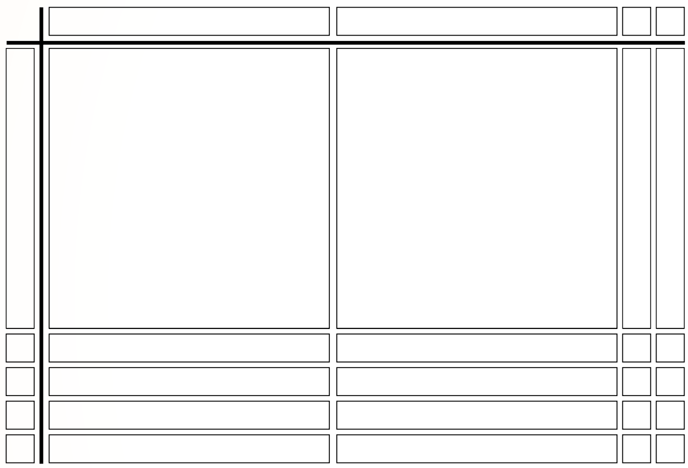
When comparing **figure 7** and **figure 13** to **figure 14**, we notice that the size of the base does not matter in the picture; so we can replace each base number rod with *b* for

FIGURE 13

The area models for multiplication in base nine are identical to the area models for multiplication in base ten.



$$(10_{nine} + 2_{nine}) \times 4_{nine} =$$
$$40_{nine} + 8_{nine} =$$
$$48_{nine} =$$



$$(20_{nine} + 3_{nine}) \times (10_{nine} + 4_{nine}) =$$
$$200_{nine} + 30_{nine} + 110_{nine} + 13_{nine} =$$
$$353_{nine}$$

base, and all of the multiplying concepts, including the distributive property, remain the same. Additionally, in standard algorithms for arithmetic, we align our columns so that we add quantities of the same place value; whereas in algebra, we combine like terms (such as shapes). Although there are many similarities, there are differences as well. There is no regrouping—for instance, ten *b*s do not make a b^2. Also, in arithmetic, we can add hundreds and tens, for example; but in algebra, we cannot combine unlike terms. Thus the “answer” becomes an algebraic expression with a term for each type of block used in the diagram. Although base blocks (or algebra blocks) become cumbersome for three-digit numbers (polynomials with terms having exponents greater than one), students should work with such examples to verify that the distributive property continues to hold.

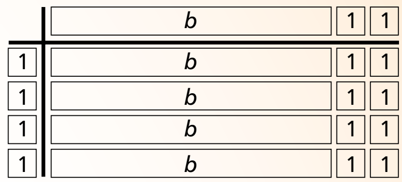
Helping students make connections like those illustrated here allows them to deepen their conceptual understanding (Baek 2008; Clements 1999; CCSSI 2010; NRC 2005) that the distributive property is not an algorithm but a *property*, a characteristic that holds throughout mathematics—arithmetic, geometry, algebra, and other branches as well. When they begin to recognize it in different contexts, this familiarity will increase their understanding, build their confidence, and increase their success in algebra and beyond (Carpenter, Franke, and Levi 2003; Kaput 1999).

Making the connections

Some educators question the appropriateness of including the distributive property at the third-grade level in the Common Core State Standards for Mathematics, wondering whether students are ready to understand it. We found that the distributive property is naturally logical to most students if we first allow students to think through problems, view them from multiple perspectives—numerically and geometrically—and then connect other models and more efficient procedures to those original models. Unfortunately, we often jump from one method to the next without making connections between the methods or to the underlying ideas that hold them all together. The value of each process or model described in this article will be undermined if students do not understand how each is connected to the previous one. And at some point, we must give our students a name for this idea—the distributive property—that keeps coming up over and over again. We must then continue to use that phrase whenever it appears in different contexts. We know we are teaching this effectively when, as students are faced with "new" problems, we hear them say, "Oh, that's just the distributive property again. I know how to do that."

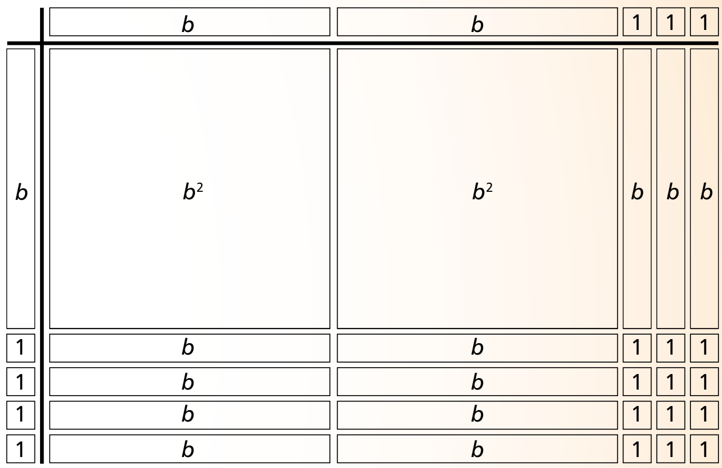
FIGURE 14

The area model for multiplication of algebraic terms resembles the area models used to multiply integers.



$(b + 2) \times 4 =$

$4b + 8$



$(2b + 3) \times (b + 4)$

$2b^2 + 3b + 8b + 12$

$2b^2 + 11b + 12$

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