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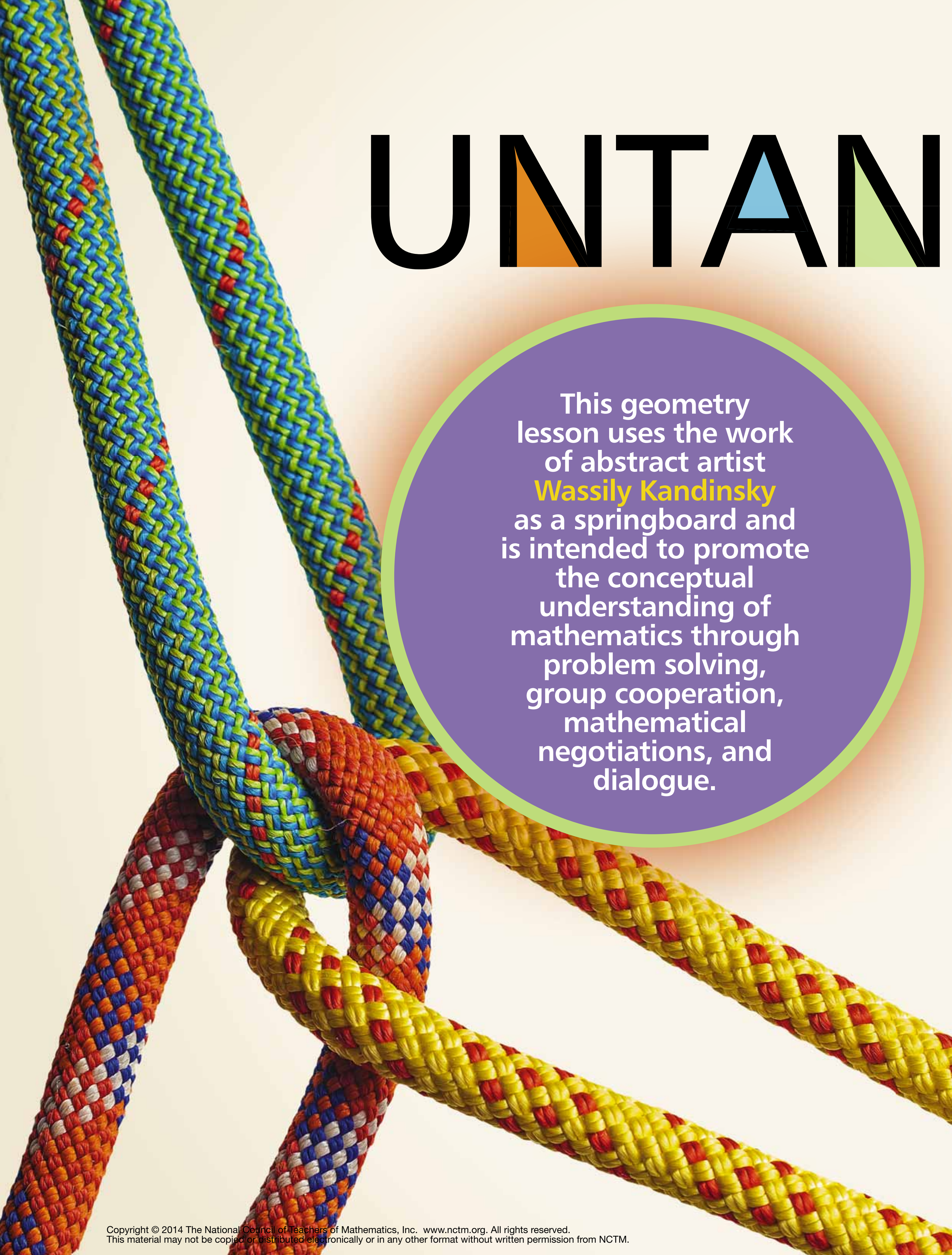

This geometry lesson uses the work of abstract artist Wassily Kandinsky as a springboard and is intended to promote the conceptual understanding of mathematics through problem solving, group cooperation, mathematical negotiations, and dialogue.

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GLING GEOMETRIC IDEAS

By
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Designed for a broad audience, including educators, camp directors, afterschool coordinators, and preservice teachers, this investigation aims to help individuals experience mathematics in unconventional and exciting ways by engaging them in the physical activity of building geometric shapes using ropes. Through this engagement, I anticipated that individuals would develop conceptual understandings that moved beyond the memorization of algorithms. After engaging physically and mentally in this activity, participants in diverse settings were able to verbally express their conceptual understandings of topics related to geometric measurement that are often difficult to grasp, such as area, perimeter, and triangle classification (Nitabach and Lehrer 1996; Thompson and Preston 2004; McDuffie and Eve 2009).

What follows are the main components of the Untangling Geometric Ideas activity and a description of how it took shape as it was used with different audiences of participants.

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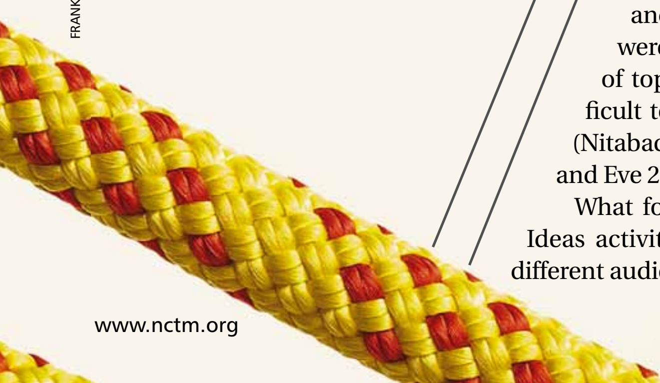
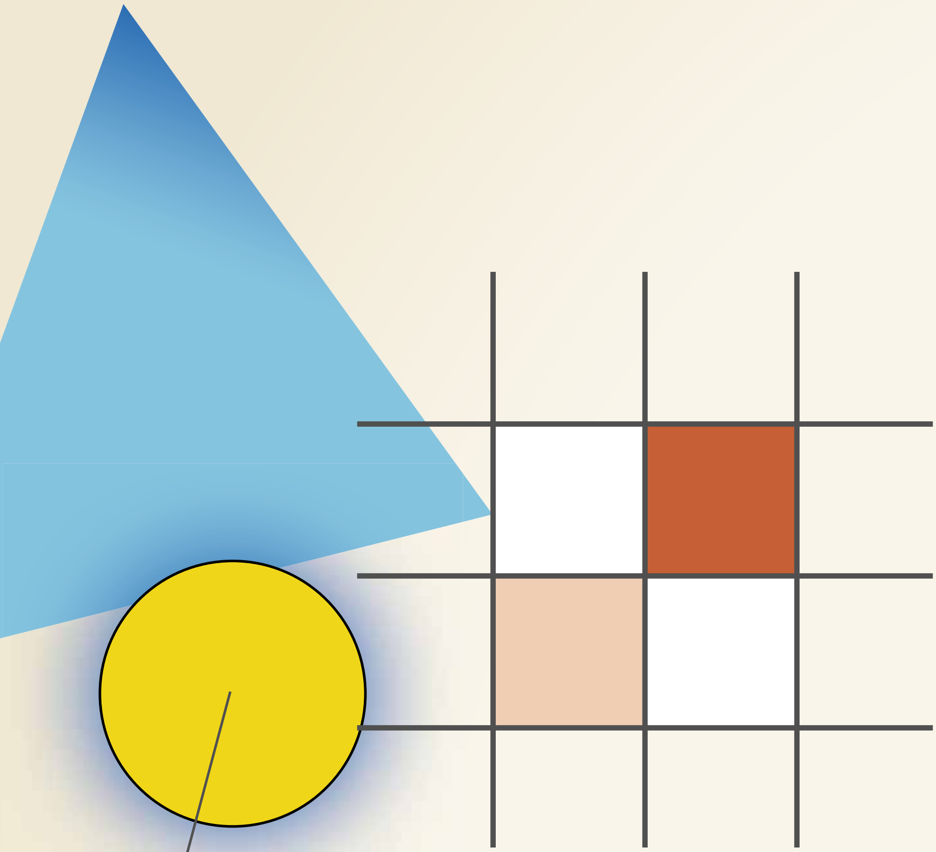




Rationale

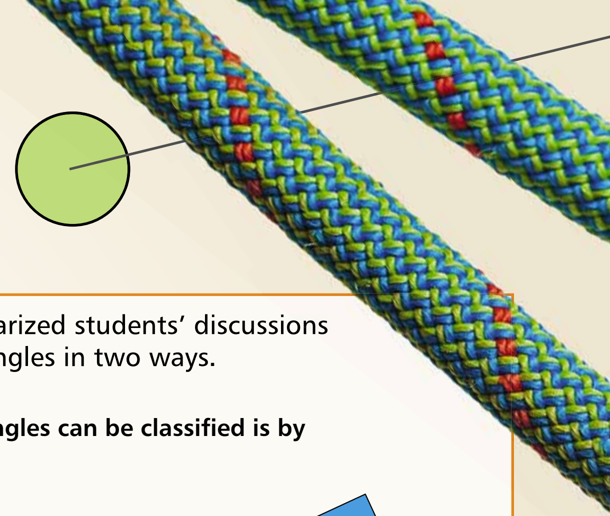
Classroom activities can influence an individual's perceptions of mathematics. Research suggests two prevalent views of mathematics. One view is algorithmically situated and solution driven; the other view is process-oriented and driven by conceptual understanding (Cobb et al. 1992; Kloosterman and Clapp Cougan 1994; Muis 2004). Each of these views has benefits, but the latter is too often ignored. For this reason, engaging students in activities that promote deep and meaningful understanding of mathematical content is imperative. Teachers of geometry frequently focus students' attention on memorizing definitions, attributes, and formulas and neglect to engage students in learning experiences that promote conceptual understanding. By focusing on rules and procedures rather than on conceptual understanding, students come to perceive the purpose of mathematical engagement narrowly (Kloosterman and Stage 1992). In contrast, the Untangling Geometric Ideas activity was designed to help educators promote the conceptual understanding of mathematics through problem solving, group cooperation, mathematical negotiations, and dialogue.

Building vocabulary

To build vocabulary before beginning the activity, participants were asked to contemplate a Wassily Kandinsky print, *Composition VIII* (1923) (http://www.guggenheim.org/new-york/collections/collection-online/artwork/1924), which was projected on a screen. Then they were to indicate how the print relates to mathematics. The print was selected purposefully as a way to "hook" individuals into thinking about mathematics and to provide an avenue of comfort for those who might otherwise experience anxiety when asked to participate mathematically.

Following this short discussion, participants were placed in groups of four, and each group was given a variety of triangle die-cuts. Each group was asked to organize the triangles by similar traits. To provide participants the freedom needed to problem solve, make sense of the mathematics, and develop unique solutions, group members were not given characteristics by which to organize their triangles, nor were they given a specific number of groups in which to categorize their triangles. In this part of the investigation, participants, in most cases, chose to categorize their triangles by size, color, number of congruent sides, or types of angles. After they were given ample time to organize and categorize their triangles, a class discussion ensued about the ways in which learners organized their triangles and the thinking that grounded their decisions. On one occasion when no groups organized their triangles by the number of congruent sides or type of angles, the activity was repeated by asking students to categorize their triangles in ways that had not yet been discussed. Once again, following the student investigation, a discussion took place that required students to communicate and justify their solutions.

At this point in the lesson, most learners did not use the mathematical terminology associated with the classification of triangles by number of congruent sides (scalene, isosceles, and equilateral) and types of angles (acute, right, obtuse, and equiangular). For this reason, introducing appropriate mathematical terminology became important when revoicing students' ideas. On one occasion when a student stated that some of the triangles had equal sides, I asked him to hold up one such equilateral triangle to show the class. On another occasion when a student talked about triangles that had similar angles, I asked her to hold up the triangle that was classified as equiangular because of its three congruent angles.



Following the discussion of triangle classification, participants received two handouts summarizing the ideas discussed (see **figs. 1a** and **1b**). After briefly reviewing the handouts, participants were asked to define and discuss area and perimeter to activate prior knowledge and to ensure that they understood the mathematical concepts needed to complete the investigation. During this discussion, as well as others, the Kandinsky print was used as a resource to help individuals clarify and communicate their mathematical ideas related to a broad array of geometric topics. The print allowed individuals to connect visual representations to their own verbalizations so that they could communicate more effectively. After receiving the handouts, students were encouraged to use the appropriate mathematical language to describe the print. In almost all cases, the appropriate language was used. One student, for example, pointed at a triangle on the print and stated that he believed it to be an equilateral triangle. Another student commented that she did not think it was equilateral because it did not look like it had sides that were exactly the same length. Another student stated that it looked like an isosceles triangle but that one would have to measure the sides of the triangle to be sure.

FIGURE 1

Two handouts summarized students' discussions about classifying triangles in two ways.

(a) One way that triangles can be classified is by their sides:

Scalene—no congruent sides

Isosceles—at least two congruent sides

Equilateral—three congruent sides

(b) Another way that triangles can be classified is by their angles:

Acute—All angles measure less than 90°.

Right—One angle measures exactly 90°.

Obtuse—One angle measures more than 90°.

Equiangular—All angles measure the same (60°). This is the same as the equilateral triangle.

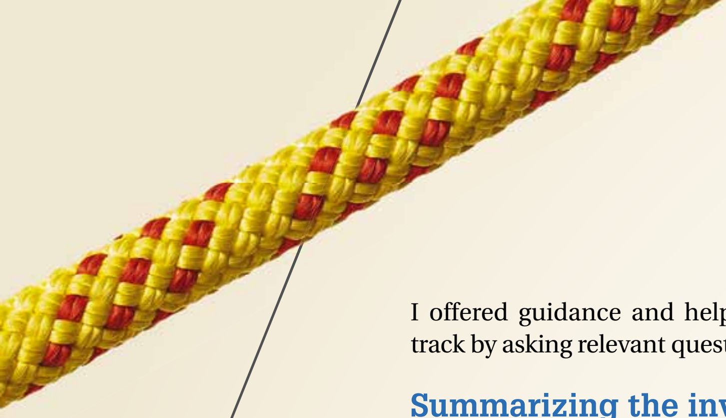
Exploring geometry through ropes

The learners were separated into groups depending on the number of participants. Groups were given 100 feet of nonstretch nylon rope (ends burned to limit fraying) that had a black permanent ink mark drawn at one-foot intervals. Each group was also given six linoleum tiles that measured 1 square foot each. In addition to these materials, each participant was given a five-page investigation activity sheet (see the **online appendix**) attached to a clipboard.

After groups were created and jobs were assigned (see the **sidebar** on **p. 512**), the groups were asked to work cooperatively to build the plane figures with the ropes and to communicate their thinking while engaging in the activity. Through the process of communication, students were expected to enhance their clarity and understanding of the mathematics of the lesson as well as their ability to effectively complete the activity sheet.

Learners were taken to a large open space where they had ample room to move around. They were directed to begin working together to build the plane figures and to fully explore the questions on the investigation activity sheet.

As a facilitator, I moved from group to group, listening to participants discuss their ideas, reasonings, and conjectures, and making sure that group members were working together in ways that promoted understanding. When necessary,



I offered guidance and helped groups get on track by asking relevant questions.

Summarizing the investigation

After all the groups had finished the investigation, they returned to the classroom to reflect on what they had learned and to allow time for the scribe to compile the information on the bottom section of the handout that asked what the group had learned from the investigation and what had surprised them mathematically (see **activity sheet, p. 5**). After finishing the task, groups shared their mathematical methods and their thoughts about the investigation. By summarizing the lesson through dialogue, participants were able to make sense of the mathematical ideas embedded in the lesson and to discuss alternative strategies. An example of alternative strategies used by different groups involved finding the rectangles with areas of 16 with the largest and smallest perimeters. One group used the ropes to build all the possible rectangles with areas of 16 and then determined the perimeters of the rectangles to identify the ones with the largest and smallest perimeter. Another group realized through discussion that the rectangle that was the longest (with a width of one unit) would have the greatest perimeter, and then the group built other rectangles to determine the rectangle with the smallest perimeter. A third group built all the rectangles but had to negotiate through discussion with one another whether a square was a rectangle and whether a square with sides measuring four units could be the "rectangle" with the smallest perimeter. By allowing individuals to actively participate in the lesson, communicate in groups, negotiate understandings, and share alternative strategies, students could focus on the process of the lesson and conceptual understandings rather than on a single correct answer.

Teacher insights

Throughout these diverse teaching and learning experiences, wonderful things came to the surface. Besides the richness of the solutions, the diversity of students' thinking, and the overall engagement of the learners, I began to see learners making connections and thinking differently about what it means to understand geometry and measurement.

Almost immediately, it became obvious that many learners were connecting geometry and measurement with arithmetic. This could be seen when many individuals initially added tiles to determine the area of the rectangles and then, without prompting, seemed to recognize that multiplying one side by another ($l \times w$) would be a more efficient means by which to generate an answer. The same was true for individuals working with perimeter. Initially, many participants added the linear units around the plane figures by counting each individually. Then they developed for themselves, and in conjunction with others, new methods for determining perimeter. Some found the sum of the length and the width of a given rectangle and then multiplied the sum by two. Others found the sum of the length multiplied by two and the width multiplied by two.

Learners talked about the ease of understanding the meanings of area and perimeter when using the ropes and 12×12 inch tiles and the idea that they believed they now understood area and perimeter rather than just having memorized definitions. Some learners stated how the formula for finding the perimeter of a rectangle ($2l \times 2w$) made more sense to them as a result of engaging with the mathematics of the investigation. Other students developed a conceptual

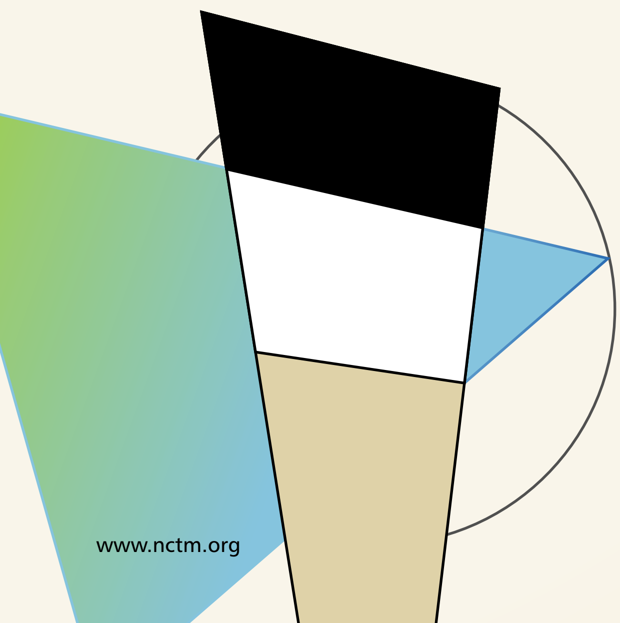
Group-member roles

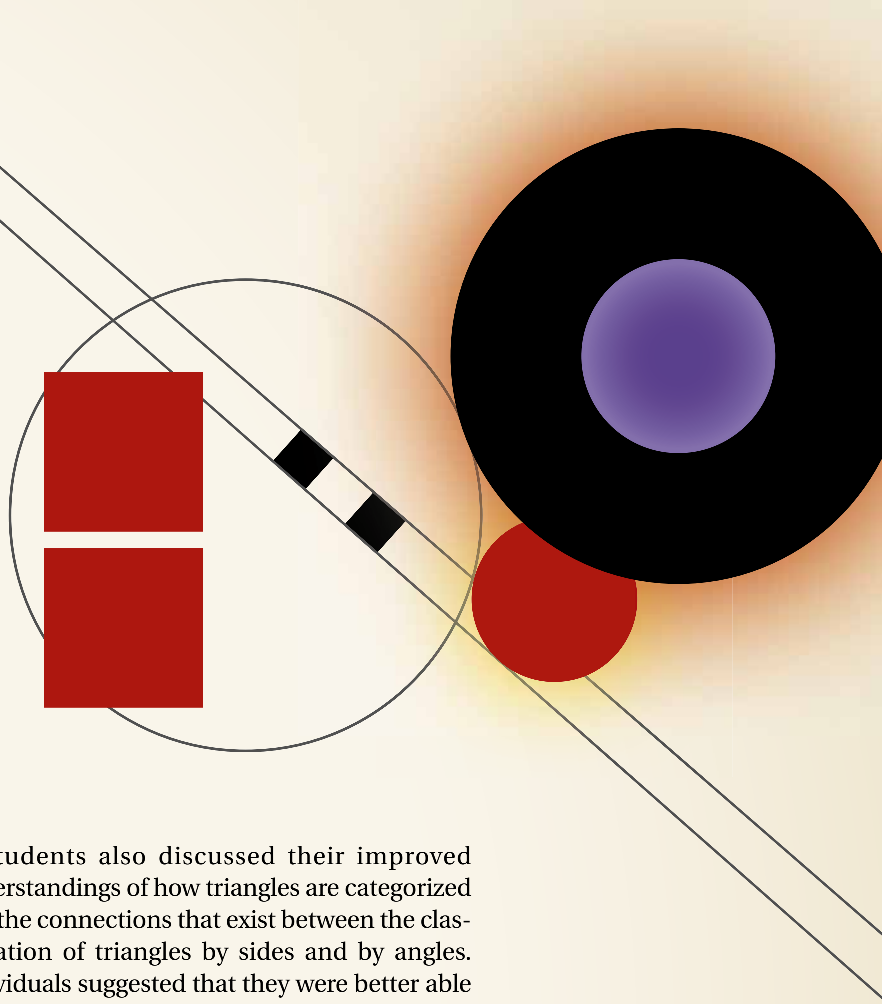
In some implementations of the investigation, participants were assigned a job or jobs:

- **Un-roper**: Responsible for getting the rope untangled and ready to use. This student was allowed to solicit help from other group members.
- **Re-roper**: Responsible for winding up the rope and bringing it back inside.
- **Tilest:** Responsible for taking out and bringing back the 12×12 inch tiles.
- **Answer examiner**: Responsible for making sure that all other group members had the answers written on their activity sheets.
- **Task master**: Responsible for keeping the group on task and helping to steer group members away from nonrelated conversations.
- **Scribe**: Responsible for writing down what the group had learned once they returned back inside the classroom.
- **Enforcer**: Responsible for making sure that *all* group members were touching the rope at all times. (The job of Enforcer ensures that no individuals take roles that are purely observational in nature and ensures that all individuals work cooperatively, without dropping the rope, to build the plane figures indicated on the activity sheet).

understanding that multiple formulas could be used for completing a single mathematical task. An example of this was when students realized that finding the perimeter of a rectangle could be accomplished by using the formula $2l \times 2w$ or the formula $2(l + w)$. Through participation and discourse, individuals realized that although both formulas generated the same product, the expressions were fundamentally different with respect to the process. Students also discussed how they made sense of the formula for area of a rectangle as $l \times w$. Some suggested that they made sense of the formula through adding the 12×12 inch tiles and recognizing a pattern that led them to multiply one side by another. Others suggested that they realized the formula by seeing the connections between adding the tiles through repeated addition and finding the area of the rectangle using the length times the width. Overall, students seemed to have a better understanding of how these formulas for perimeter and area were derived.

Some individuals suggested that following this investigation, students may have an easier time understanding that the area of a triangle is one half the area of a rectangle, rather than the formula $1/2bh$. Others suggested that after this activity, students may even conceptualize the algorithm as finding the area of a rectangle that would surround a given triangle and then dividing the area of that rectangle into two equal parts. Once again, through their own conceptual understandings and mathematical processes, individuals began to recognize how multiple formulas could be used to generate equivalent answers.





Students also discussed their improved understandings of how triangles are categorized and the connections that exist between the classification of triangles by sides and by angles. Individuals suggested that they were better able to visualize the classifications after the investigation as opposed to relying on the memorization of the definitions.

I was surprised when multiple groups spontaneously connected the idea of teaching multiplication facts and the activities within the investigation. On more than one occasion, learners stated how they believed that engaging students in the building of the rope rectangles would help them understand their multiplication facts because the area of the rectangles created by the ropes (1 × 16 ft., 2 × 8 ft., 4 × 4 ft.) could help students visually understand why 1 × 16 is 16, 2 × 8 is 16, and 4 × 4 is 16. What the participants were describing is the area model of multiplication often taught through the building of rectangular arrays. Although this may appear to be a simplistic connection, students were connecting this activity with the teaching of multiplication facts, which is often done through rote memorization, weekly tests, flashcards, and competition.

Many learners commented that when they were in school, they learned algorithms first and then were asked to solve problems using those algorithms. Some learners seemed angered at the idea that they had not been taught to understand these ideas years earlier, and they seemed emotionally frustrated that they had been denied the opportunity to engage with what

MANY LEARNERS RECOGNIZED THAT THE FORMULA DID NOT HAVE TO BE EXPLICITLY TAUGHT BEFORE A MATHEMATICAL INVESTIGATION.

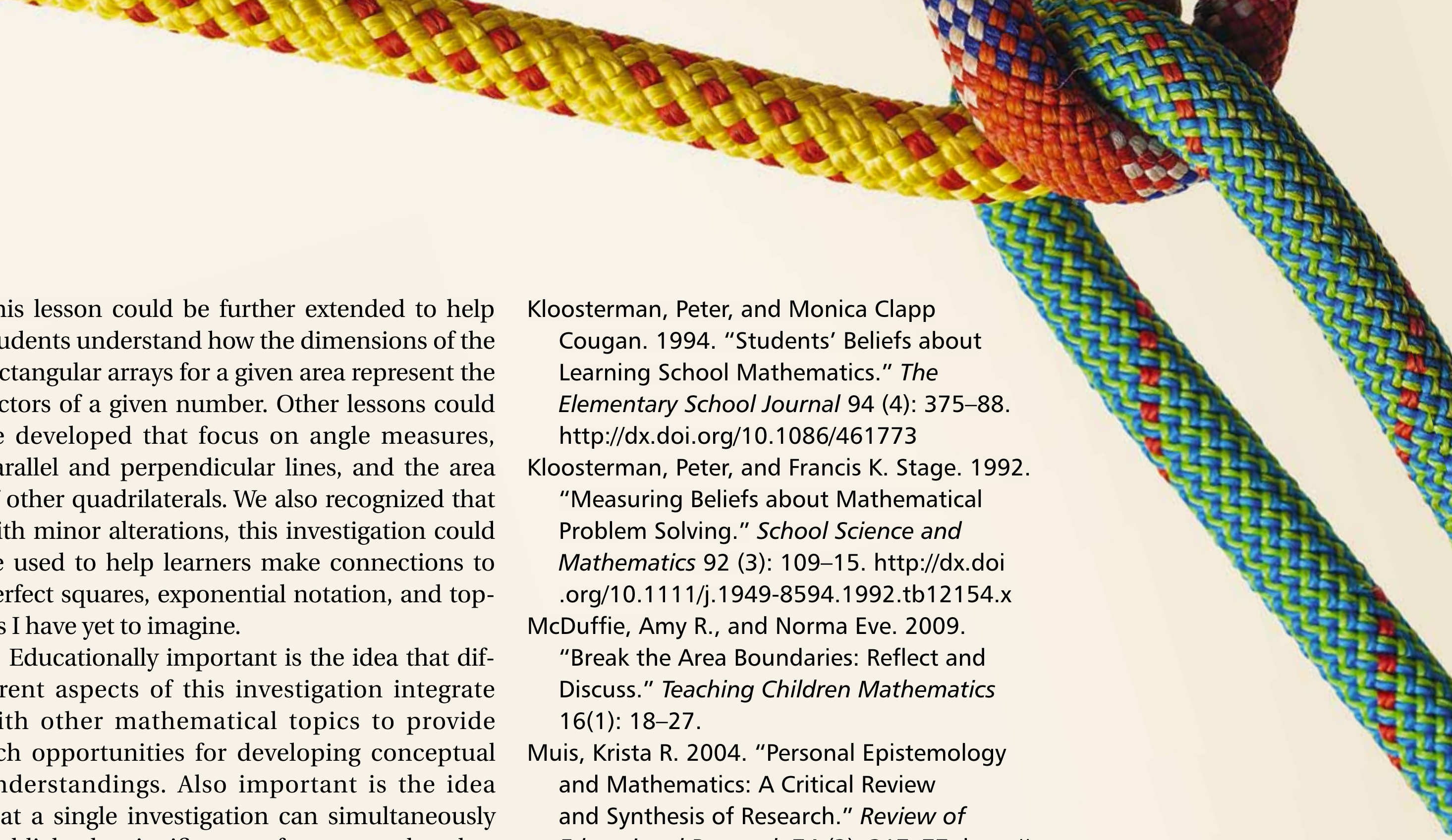
seemed to them to be a relatively simple investigation that led to the deep understanding of geometric concepts. As the facilitator, I thought it important at this point to help learners reflect on the idea that learning mathematics is not about investigations or algorithms alone, but rather about meaningful mathematical experiences that allow learners to develop meaningful connections between the two. It was also essential to reiterate the importance of conceptual understanding and its relationship to participatory investigations.

Throughout this activity, a multitude of comments made it obvious that many learners recognized that the formula did not have to be explicitly taught before engaging students in a mathematical investigation. They seemed to recognize that students at different developmental levels and with dissimilar levels of formal, mathematical knowledge could engage in meaningful mathematical experiences when the investigation was contextualized in such a way as to engage students in what Fosnot and Dolk (2002) refer to as “problem solving and problem posing” (p. 31). Many who participated in this investigation recognized the power of allowing students to discover their own mathematical meanings for algorithms and mathematics in general. These participants seemed to notice the power of what Fosnot and Dolk refer to as *mathematizing* or, in other words, allowing children to “explore situations mathematically” (p. 9) by way of investigating patterns, making conjectures, communicating mathematically with others, and attempting to make sense of mathematics through the use of mathematical models.

Helping educators, both current and future, understand that an algorithm need not drive a mathematical activity is often a necessary challenge. That is the power of this investigation within the teaching-learning context. What I witnessed from learners who participated in this activity was their willingness to begin considering the notion that well-thought-out, mathematical investigations can help students not only discover algorithms but also conceptually understand algorithms in the process. These learners experienced firsthand the power of learning through investigations and personal discovery. They gained an appreciation for the difference between what it means to *know* and what it means to *understand* mathematics. Although they may have read about or thought about these ideas, their comments indicate that they may have previously undervalued them.

The future of ropes

This mathematical investigation was designed to offer participants an opportunity to engage in an investigation that would help them see and feel the benefits of investigative mathematics. It was hoped that through their experiences, participants would find ways to replicate the investigation in a variety of settings, in and out of mathematics classes, to enhance the mathematical experiences of others. We recognized that this activity could be used as a basis for developing new lessons that focus on understandings. One such lesson might involve having students build rectangular arrays with ropes and then guiding them to discover the area model of multiplication and its connection to repeated addition. Another lesson might involve having students build all the rectangular arrays for a given area and then helping them visualize the Commutative Property of Multiplication as equivalent rectangles that are rotated 90 degrees and the Identity Property of Multiplication as rectangles that have a width of one and an area that is equivalent to the length of the other side.



This lesson could be further extended to help students understand how the dimensions of the rectangular arrays for a given area represent the factors of a given number. Other lessons could be developed that focus on angle measures, parallel and perpendicular lines, and the area of other quadrilaterals. We also recognized that with minor alterations, this investigation could be used to help learners make connections to perfect squares, exponential notation, and topics I have yet to imagine.

Educationally important is the idea that different aspects of this investigation integrate with other mathematical topics to provide rich opportunities for developing conceptual understandings. Also important is the idea that a single investigation can simultaneously highlight the significance of conceptual understandings and enhance student engagement, discovery learning, communication, reasoning, cooperative group work, and mathematical dialogue. This investigation and others like it build bridges between engaging mathematically and deeply understanding mathematical content. That being said, I am hopeful that educators will continue to develop meaningful mathematical investigations that align with a variety of mathematical standards. Such alignments will show more and more clearly that understanding mathematics is not about investigations or standards but rather about the amalgamation of the two.

Common Core Connections

3.MD.C.5 **3.MD.C.7A**
3.MD.D.6 **3.MD.C.7B**

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