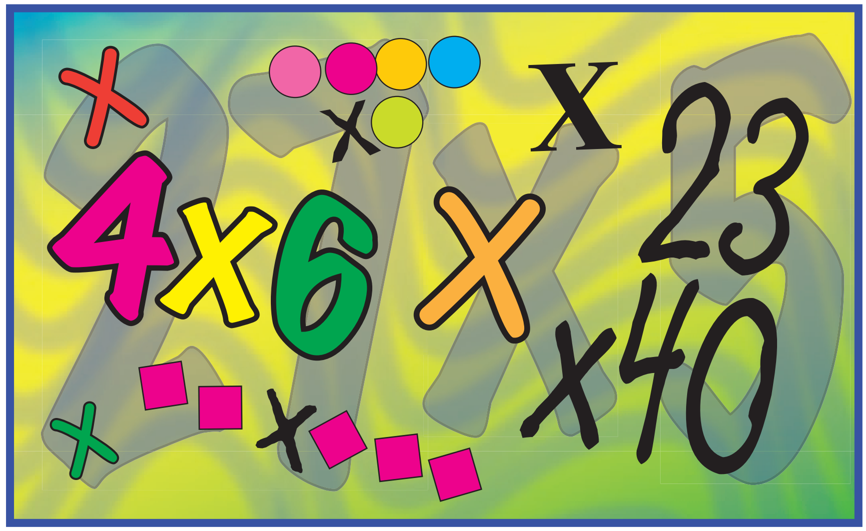
Student Explorations in Mathematics

Demystifying Multiplication



This activity begins with a brief formative assessment of student understanding of multiplication of two-digit numbers. It supports students' understanding of multiplication and its application. Constructivist pedagogy is used to move learning from the concrete (using manipulatives) to the representational (using pictures and diagrams) to the abstract (using algorithms). Students build models of the operation 27 × 15 and its result in a variety of ways. The activity promotes student reasoning and sense making by analyzing various multiplication algorithms (area models, partial products, lattice multiplication, and the traditional method).

This activity is an enhancement of the *Student Math Notes* (*SMN*) activity "The Marvels of Multiplication" from November 2008. A 2010–2011 user survey identified the original activity as one of the most popular *SMN* activities in the current collection of over eighty-five activities available at **www.nctm.org/sem** (click "Back Issues"). On the basis of this feedback, the activity was selected for a "face-lift" and now includes suggested solutions, teaching notes, ideas for integration with art and literature, and recommended technology applications.

TEACHER NOTES

Suggested answers are in **red**. Instructional notes are in **blue** and are preceded by the hand icon. These two features do not appear in the Student Edition.

Classroom materials

- Markers and paper
- A classroom set of at least one of the following manipulatives: base-ten blocks, algebra tiles, graph paper cut into strips and squares, or Cuisenaire® rods
- Grid paper

Optional materials

- Check the NCTM Illuminations website Algebra Tiles: http://illuminations.nctm.org/ActivityDetail.aspx?ID=216

Demystifying Multiplication

Multiplication is one of four basic arithmetic operations. It is often seen as a table of facts to memorize, but how multiplication actually works is a mystery to some people. The purpose of this exploration is to understand the meaning behind multiplication and eliminate the mystery.

Getting Started

Think of some examples of multiplication use in everyday life.

Ask students, "When, in real life, do you use multiplication?" You may want to provide an example to start, such as buying items for a birthday party. Distribute sets of manipulatives to individual students (such as base-ten blocks, algebra tiles, graph paper cut into strips and squares, or Cuisenaire® Rods).

Part 1. Using Manipulatives to Investigate Multiplication

1. Use manipulatives to demonstrate what 3×4 looks like.

Answers may vary. Students should display 12 tiles or other items on their desk.

Some students may form a rectangular array with dimensions 3×4. Others may just make a pile of 12 items or 3 piles of 4 items on their desks. Encourage students to organize the items in arrays.

2. Use manipulatives to demonstrate what 4×3 looks like.

Answers may vary. Students should display 12 items on their desk.

Students may again form rectangular arrays or 4 piles of 3 items. Students should notice that the 2 arrays represent the same product and have the same answer.

3. What do you notice about 3×4 and 4×3?

Answers may vary but should reflect that their models of 3×4 and 4×3 represent the same quantity.

The 2 arrays that represent the same product is an example of the commutative property. Discuss with students when the order of numbers matters and when it does not. In the area models throughout this activity, the first factor in the product is sometimes the vertical dimension of the rectangle and at other times is the horizontal dimension. Call students' attention to this fact if they do not notice it themselves.

Using Tiles to Learn to Represent Larger Numbers

Move students into small groups and supply them with paper and markers. They should bring their individual sets of manipulatives to the group that they join.

Formative Assessment

Begin by having students share their thinking about the meaning of 27 × 15. Probe their thinking as to what they mean by words like *times* and *multiply*. You are looking for the understanding that 27 × 15 means 27 groups, or sets, of 15. Record students' ideas and comments on the board.

4. What is the product of 27 × 15?

405

5. Use your group's manipulatives to model 27 × 15.

Students should form a rectangular array with dimensions 27 and 15 with a total of 405 tiles. Answers may vary.

Be sure students have enough manipulatives to complete the activity successfully. Emphasize that you are not looking for them to answer the problem by working out the solution but to model the meaning. Encourage students to use graph paper and shade in boxes.

Differentiation by Content

Use alternate problems for varying levels of understanding (7 × 5, 12 × 8, etc.). Some suggested additional student questions include the following:

- What ways can you think of to use the tiles to represent the product?
- How does this compare with 27 × 15?
- Are they the same or different?
- How can you prove they are the same or different?

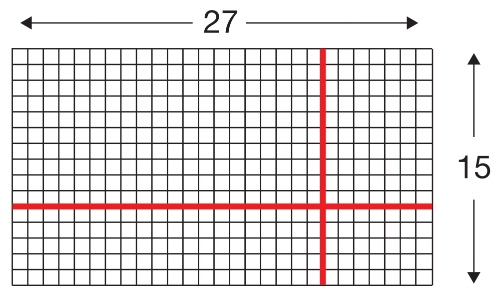
Students should be able to view the tiles as looking different but representing the same area if they are rotated. If students use a hundred block to represent their answer, ask how their model of tiles shows 27 × 15 and not just 405 separate blocks. Guide them toward the geometric model.

Part 2. Marvels of the Geometric Model

You can write the number 15 in expanded notation as the sum of 1 ten (10) and 5 ones (5). In the same way, you can also write the number 27 as the sum of 2 tens (20) and 7 ones (7).

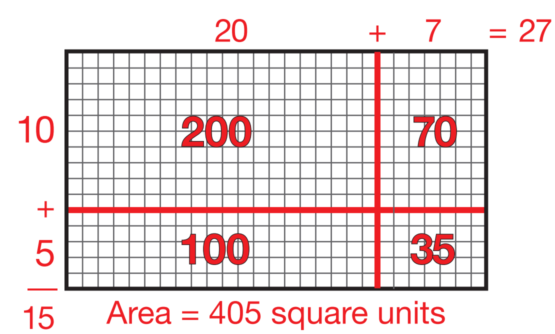
Draw lines on the 15 × 27 grid in the next column to separate the length into sections of 20 + 7 and the width into sections of 10 + 5.

Although this question seems the same as the fourth question in Part 1, this strategy employs a method that can be used to multiply binomials and links it to a method that students know how to use.



Area = 405 square units

6. Find the areas of each of the four sections in the rectangle.

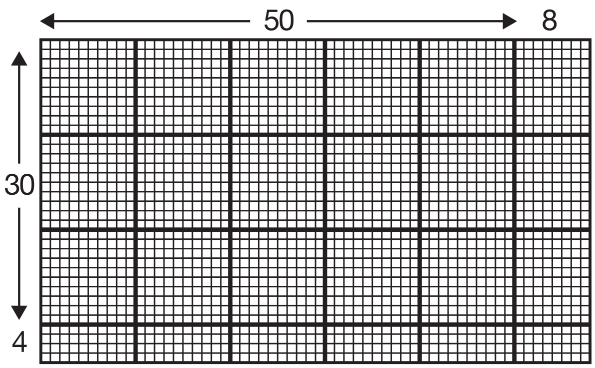


7. Find the sum of these areas. Compare this sum to the answer you obtain when multiplying 15 × 27.

(10 × 20) + (5 × 20) + (10 × 7) + (5 × 7) = 200 + 100 + 70 + 35 = 405 square units. This is the same answer as 15 × 27 = 405.

The area of each smaller rectangle is a part of the area of the larger rectangle. In other words, each section of the geometric model is a *partial product* of the whole product of 15 × 27.

8. The figure below is a geometric model for finding the area of a rectangle. Using this model, first calculate the areas of the smaller rectangles (in bold), then add them to find the area of the large rectangle.



Sum = ________

Student answers may vary. Students may count congruent rectangles and group them to find the area. A sample answer might be (30 × 50) + (4 × 50) + (8 × 30) + (8 × 4) = 1972 square units.

9. How does the sum of the areas compare to the product of 34 and 58?

They are the same: Each one equals 1972 square units.

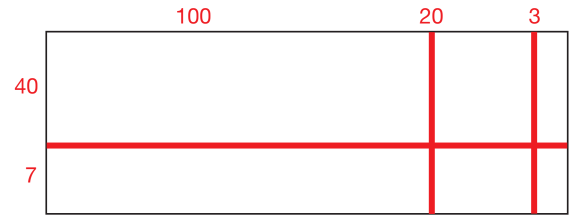
10. Use grid paper to draw a geometric model of a rectangle with a length of 65 and a width of 17. Find the partial products and the sum of those products.

Student methods may vary. Students may divide the rectangle into sections, possibly (60 + 5) and (10 + 7). The sum of these areas would be (60 × 10) + (60 × 7) + (5 × 10) + (5 × 7) = 1105 square units.

Although the dimensions in the rectangles are not always drawn to scale and in some cases are not proportional, you can still sketch the model to clarify and demonstrate your thinking.

11. Now model 47 × 123 using a geometric model. Break apart the numbers to make them easier to work with.

Student methods may vary. Some students may choose to use the product, (40 + 7)(120 + 3). A sample answer appears in the next column.



12. Using your model, what is the product of 47 × 123?

If students use this model, they may find the following areas: (40 × 100) + (7 × 100) + (40 × 20) + (7 × 20) + (40 × 3) + (7 × 3) = 5781 square units

Marvels of the Distributive Property

The geometric models you have been using are examples of the *distributive property*. When you multiply 7 × 108, you are multiplying 7(100 + 8). That can be written as 7 × 100 + 7 × 8.

13. What is the product of 7 × 108? Find the answer mentally. Record your answer.

756

14. Use the *distributive property* to find the product of 6 × 123 mentally. First think of 123 as 100 + 20 + 3. Then find 6(100 + 20 + 3).

600 + 120 + 18 = 738

Sometimes it is easier to multiply two factors when one factor is represented as the difference of 2 numbers. For example, when considering the product of 4 and 97, you can think of 97 as 100 – 3 or as 90 + 7. The expressions 4(100 – 3) and 4(90 + 7) are equivalent expressions. Therefore, you can find 4(100 – 3) or 400 – 12. You can also find 4(90 + 7) or 360 + 28. Either way, the product of 4 × 97 is 388.

Use the *distributive property* to find the following products mentally in more than one way.

15. 7 × 31

Student methods may vary. A sample answer might be 7(30) + 7(1) = 217.

16. 8×85

Student methods may vary. A sample answer might be $8(80) + 8(5) = 680$. If students express 85 as $(90 - 5)$ instead, the solution may look like $8(90) - 8(5) = 680$.

This question could initiate a rich discussion about which method is better, using a sum or using a difference to solve the question. Either is equally acceptable, but sometimes one may be better than the other. Students should be encouraged to discuss their preferences.

17. 6×299

Student methods may vary. A sample answer might be: $6(200) + 6(90) + 6(9) = 1794$. Another possible solution using difference is $6(300) - 6(1) = 1794$.

18. 16×231

Student methods may vary.

This question could also initiate a rich discussion because a variety of choices exist for students:

$16(230) + 16(1)$
$15(231) + 1(231)$
$10(231) + 6(231) = 10(231) + 6(230) + 6(1)$

to name a few.

Part 3. Using Geometric Models with the Distributive Property

In the questions above, the *geometric models* are still useful to the multiplication. When the expression uses subtraction, think of subtraction as adding negative numbers and use partial products. $2(x - 4)$ can be written as $2(x + (-4))$. Negative numbers can still be used in the geometric model, but they do not represent real lengths.

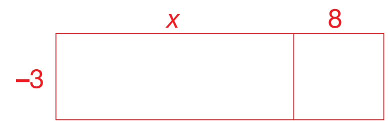
Ask students, "Can length and area be negative numbers? Why or why not?" Discuss with them that although lengths cannot be negative, the model still assists students in using the Distributive Property to answer questions.

	x	(-4)
2	$2x$	(-8)

The product of $2(x-4)$ is $2x - 8$.

19. Use a geometric model similar to the one above to evaluate $-3(x + 8)$.

Students may use a drawing similar to this model, but answers may vary.

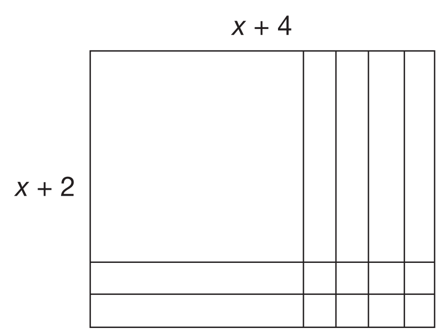


$-3x + -24 = -3x - 24$

Students' geometric models may vary considerably. The purpose of having them draw geometric models was not to have students spend a lot of time drawing an accurate picture but rather to have them use the model as a method of problem solving.

20. Evaluate $2x(x - 10)$.

$2x(x) - 2x(10) = 2x^2 - 20x$



21. The dimensions of the rectangle above can be written as $(x + 2)$ and $(x + 4)$. In the geometric model above, can you find the x^2 piece, which has the shape of a square and has an area of x^2?

Students identify the x^2 piece.

a. Find the rectangles that have an area of x. How many are there?

Students identify 6 pieces that have dimensions of 1 and x.

b. Find the rectangles that have an area of 1. How many are there?

Students identify 8 pieces that have dimensions of 1 and 1.

c. The product of the dimensions equals the area, or $(x + 2)(x + 4) = x^2 + 6x + 8$.

Students identify the entire rectangle.

Due to the limitations of the F.O.I.L. strategy for multiplying two binomials, we have chosen not to include it as part of this activity. We leave it to the discretion of the teacher whether to introduce this topic as part of a lesson or in response to student questions.

Notice that the product of $(x)(x)$ is x^2 and can represent a square. That is why x^2 is read as "x squared." The four resulting products, $x^2 + 4x + 2x + 8$, can be simplified to $x^2 + 6x + 8$ after combining like terms. The rectangle above, therefore, has an area of $x^2 + 6x + 8$.

The geometric models in the sample solutions for questions 22 and 23 are not drawn to scale. They are provided only as examples of what students may create. Because these models are used only as a tool, the dimensions need not be drawn accurately.

22. Use a geometric model to evaluate the product of $(x + 8)(x + 12)$.

	x	12
x	x^2	$12x$
8	$8x$	96

$x^2 + 12x + 8x + 96 = x^2 + 20x + 96$

You can also use geometric models to organize the process of multiplying. This is especially useful when factors are negative or include variables, where the area analogy may not be appropriate.

23. Evaluate the product of $(x - 3y)(2x + 6y)$.

	x	$-3y$
$2x$	$2x^2$	$-6xy$
$6y$	6xy	$-18y^2$

$x^2 + 6xy - 6xy - 18y^2 = 2x^2 + 18y^2$

In fact, even when you multiply two trinomials, such as $(x^2 - 5x + 7)$ and $(x^2 + 3x + 5)$, you can think of partial products and model the multiplication with the geometric model below.

	x^2	$3x$	5
x^2	x^4	$3x^3$	$5x^2$
$-5x$	$-5x^3$	$-15x^2$	$-25x$
7	$7x^2$	$21x$	35

24. Fill in the partial products in each section of the model above.

All answers in each row are provided in order, left to right.

First row: x^4, $3x^3$, $5x^2$

Second row: $-5x^3$, $-15x^2$, $-25x$

Third row: $7x^2$, $21x$, 35

25. Now write the products as a simplified sum.

$x^4 + 3x^3 - 5x^3 + 5x^2 - 15x^2 + 7x^2 - 25x + 21x + 35 =$
$x^4 - 2x^3 - 3x^2 - 4x + 35$

Encourage students to provide the expression in order of decreasing exponents.

26. Evaluate the product of $(x^2 + 4x + 10)(x^3 - 2x^2 - 5x + 9)$.

$x^5 + 2x^4 - 3x^3 - 31x^2 - 14x + 90$

Can you …

- use a model to factor the polynomial $x^2 + 5x + 6$?
- use a geometric model to show (5.2)2?
- represent question 14 using algebra tiles or base-ten blocks?
- multiply the polynomials below?

  $(x^2 + 4x + 1)(2x^2 - 5x + 9)$ or

  $x^2 + 4x + 1$ by $2x^2 - 5x + 9$
- use a geometric model to evaluate the product of mixed numbers, such as $3\frac{1}{4} \times 2\frac{1}{2}$?

Did you know that …

- when geneticists use Punnett squares, they are using this type of model?
- you can multiply 8×12 by multiplying $(10 - 2)$ and $(10 + 2)$?
- John Napier (1550–1617), a Scotsman who was famous for inventing logarithms, used a system of multiplication to construct a series of rods to help with long multiplication? These rods became known as Napier's bones.
- in a geometric model for multiplication, the dimensions are the factors and the area is the product?
- Arthur Benjamin can multiply very quickly? See http://www.ted.com/talks/arthur_benjamin_does_mathemagic.html

Resources

Pickreign, Jamar, and Robert Rogers. "Do You Understand Your Algorithms?" *Mathematics Teaching in the Middle School* 12 (August 2006): 42–47.

Tent, Margaret. "Understanding the Properties of Arithmetic: A Prerequisite for Success in Algebra." *Mathematics Teaching in the Middle School* 12 (August 2006): 22–25.

"Teaching to the Big Ideas." 1997. In *Developing Mathematical Ideas, Module 2: Making Meaning for Operations*. Newton, MA: Education Development Center.

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