

Mathematical Modeling: The Core of the Common Core State Standards

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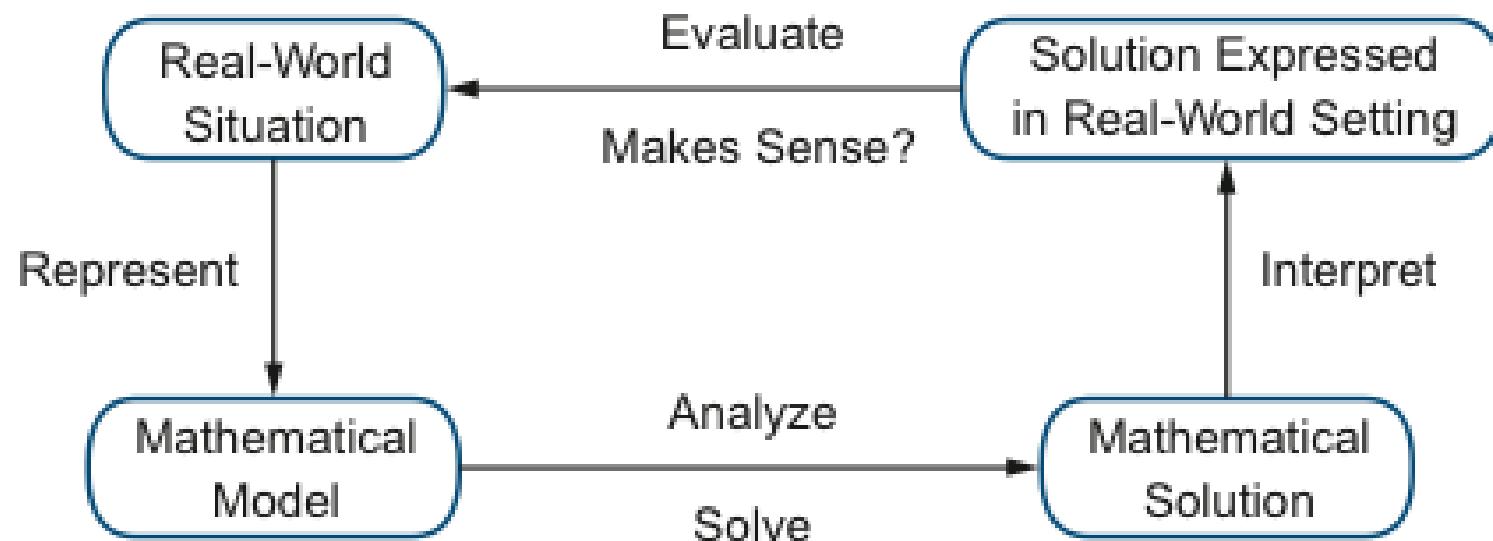
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Session Overview

- The nature and importance of mathematical modeling in high school mathematics
- Examples of modeling problems that both motivate and focus on CCSSM Content Standards and Mathematical Practices
- Participant comments and questions



Process of Mathematical Modeling



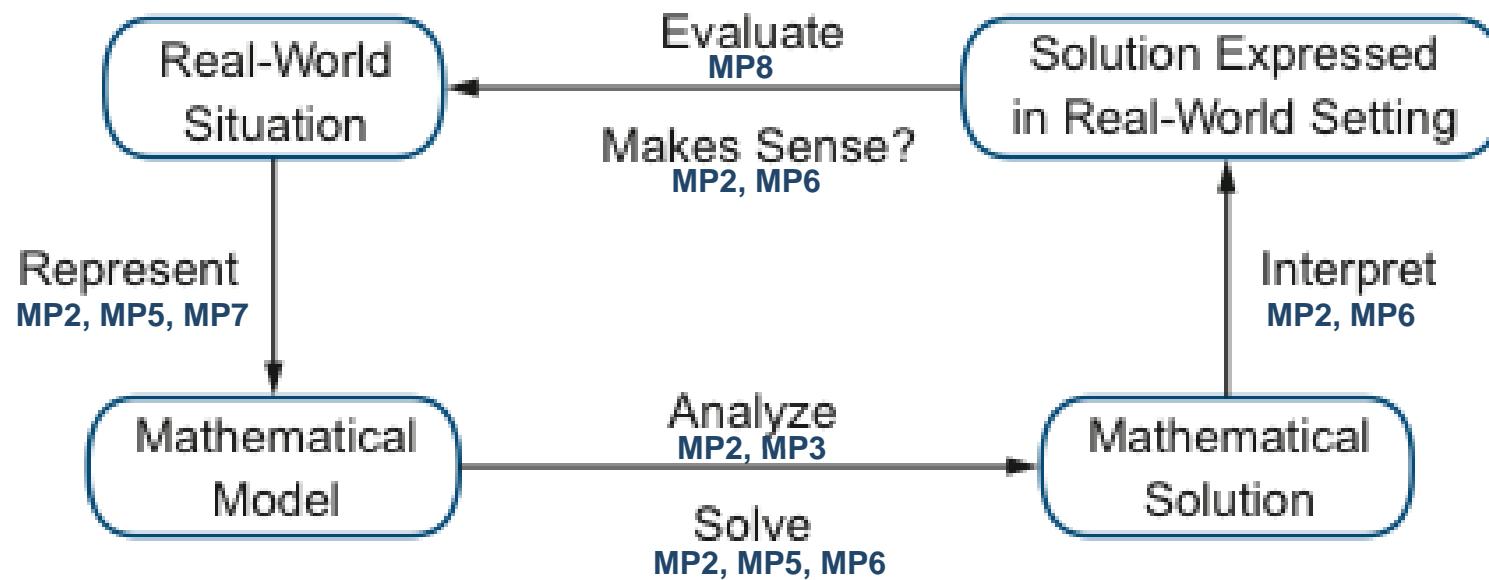
Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.



Process of Mathematical Modeling

Connecting Mathematical Practices (MP) and Content Standards (CS)
MP1 and MP4 are the focal practices of the entire process.



Translating the CCSSM into practice will require “meaningful curriculum organizations that are problem-based, informed by international models, connected, consistent, coherent, and focused on both content and mathematical practices. These new models should exploit the capabilities of emerging digital technologies ... with due attention to equity.”

Confrey & Krupa
A Summary Report from the Conference
“Curriculum Design, Development, and Implementation
in an Era of Common Core State Standards,” 2010



Modeling and Making Mathematics Problematic

- Problems are identified in context.
- Problems are studied through active engagement.
- Conclusions are reached as problems are (at least partially) resolved.
- The benefits lie not only in the solutions to the problems, but the new relationships that are discovered.

(Dewey 1929, 1956; Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier, & Wearne, 1996)



CLIMBING WALL BUSINESS PROSPECTS

The *Galaxy Sport and Outdoor Gear* company has a climbing wall in the middle of its store. Before the store opened for business, the owners needed to determine the price to charge per climb.



Getting Started

Using survey methods and linear regression, students at Lakeview High School estimated that the daily number of climbing wall customers would be related to the price per climb x by the linear function $n(x) = 100 - 4x$.

- a. According to this function, how many daily climbing wall customers will there be if the price per climb is \$10? What if the price per climb is \$15? What if the climb is offered to customers at no cost?

$$n(10) = 60$$

$$n(15) = 40$$

- b. What do the numbers 100 and -4 in the rule for $n(x)$ tell about the relationship between climb price and number of customers?
 $-4 \rightarrow$ lose 4 customers for every dollar increase in climb price

- c. Determine a function $I(x)$ that tells how daily income from the climbing wall depends on price per climb.

$$I(x) = x(100 - 4x) = 100x - 4x^2$$



The function $e(x) = 2x + 150$ shows how daily operating expenses for the Galaxy Sport climbing wall depend on the price per climb x .

- a. Write two algebraic rules for the function $P(x)$ that gives daily profit from the climbing wall as a function of price per climb

- (1) one that shows how income and operating expense functions are used in the calculation of profit, and

$$P(x) = (100x - 4x^2) - (2x + 150)$$

- (2) another that is in simpler equivalent form.

$$P(x) = -4x^2 + 98x - 150$$

- b. Find $P(5)$. Explain what this result tells about climbing wall profit prospects.

$$P(5) = 20 \text{ Price climb at } \$5 \text{ yields profit of } \$240$$

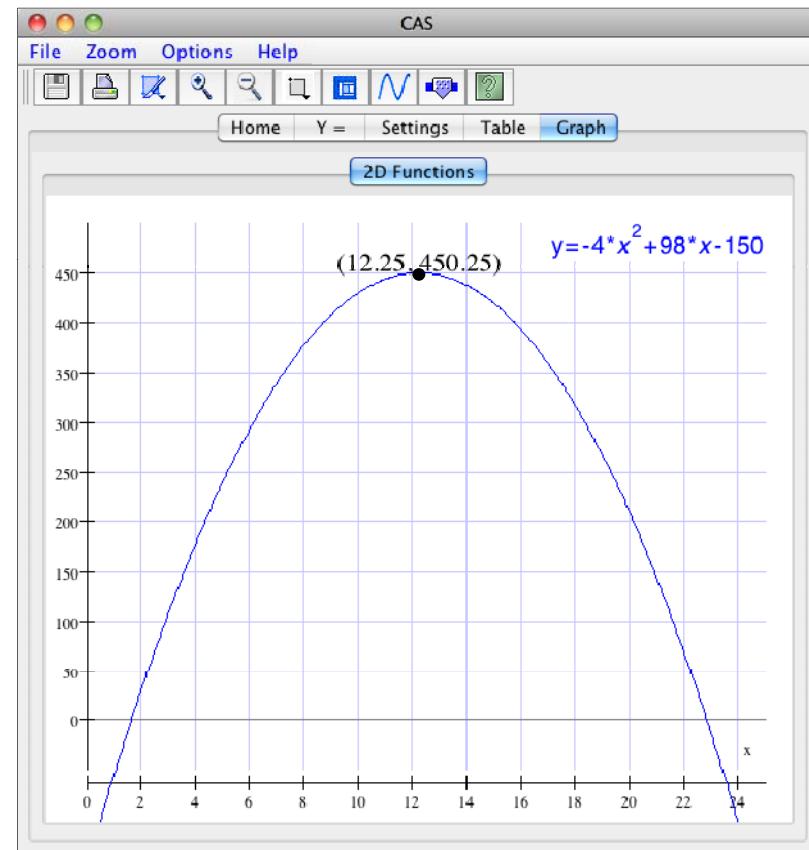
- c. Write and solve an inequality that will find the climb price(s) for which *Galaxy Sport and Outdoor Gear* will not lose money on operation of the climbing wall.

$$-4x^2 + 98x - 150 \geq 0 \text{ is true when } 1.64 \leq x \leq 22.86$$

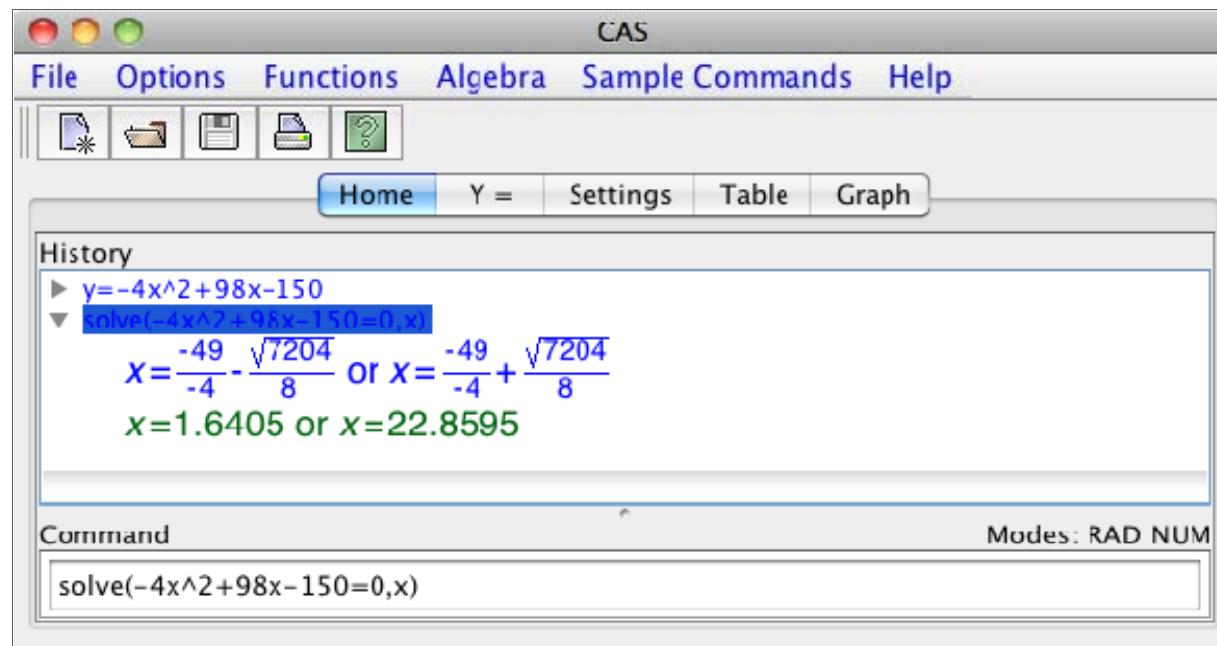


Climbing Wall Bottom Line

- a. What climbing price(s) will yield maximum daily profit from the climbing wall?



- b. For what price(s) will the climbing wall business break even?



CAS

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Home Y = Settings Table Graph

History

▶ $y = -4x^2 + 98x - 150$

▼ $\text{solve}(-4x^2 + 98x - 150 = 0, x)$

$x = \frac{-49 - \sqrt{7204}}{-4}$ or $x = \frac{-49 + \sqrt{7204}}{-4}$

$x = 1.6405$ or $x = 22.8595$

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solve(-4x^2+98x-150=0,x)



OPTIMAL REFINERY LOCATION

Drilling teams from oil companies search around the world for new sites to place oil wells. Increasingly, oil reserves are being discovered in offshore waters.

The Gulf Oil Company has drilled two high-capacity wells in the Gulf of Mexico about 5 km and 9 km from shore.

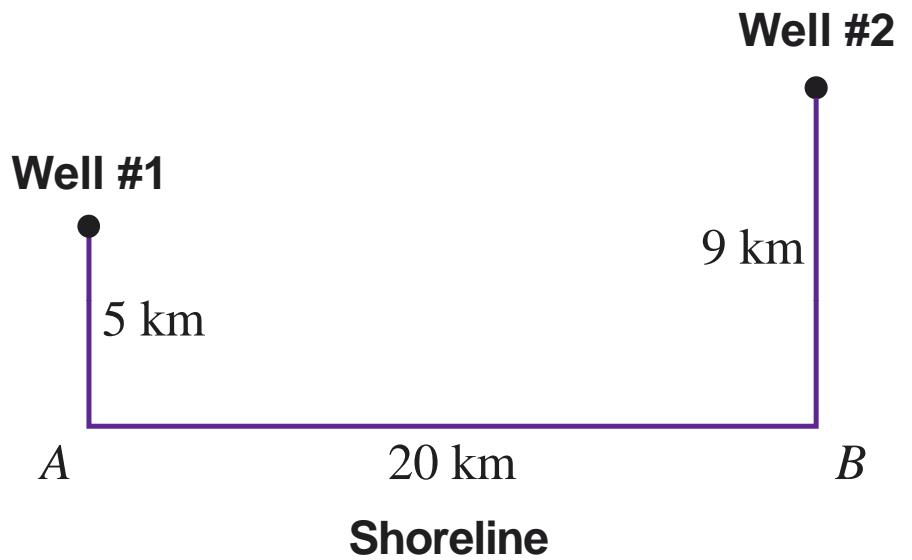


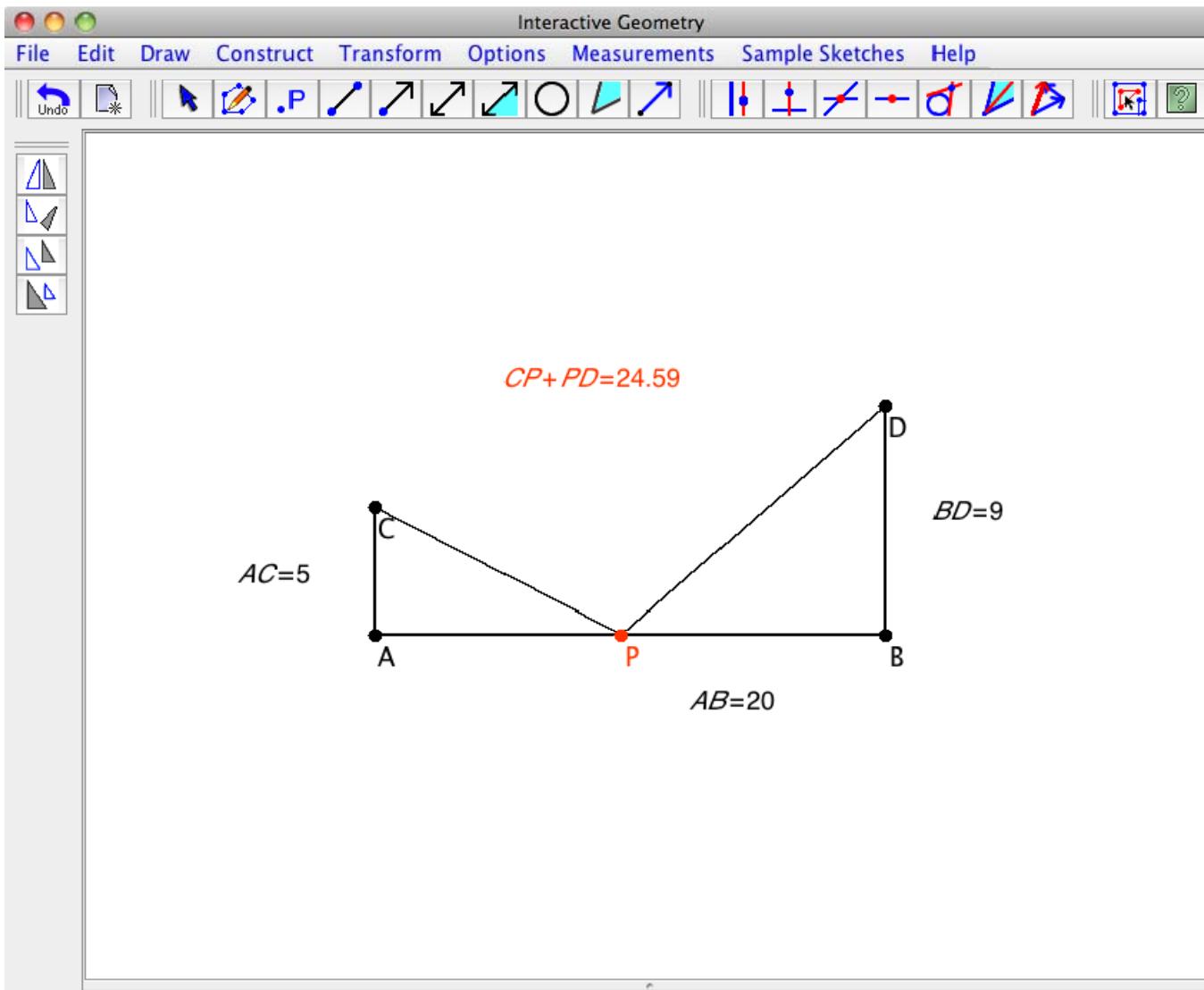
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The company wants to build a refinery to pipe oil from the two wells to a single new refinery on shore. Assume the 20 km of shoreline is nearly straight.

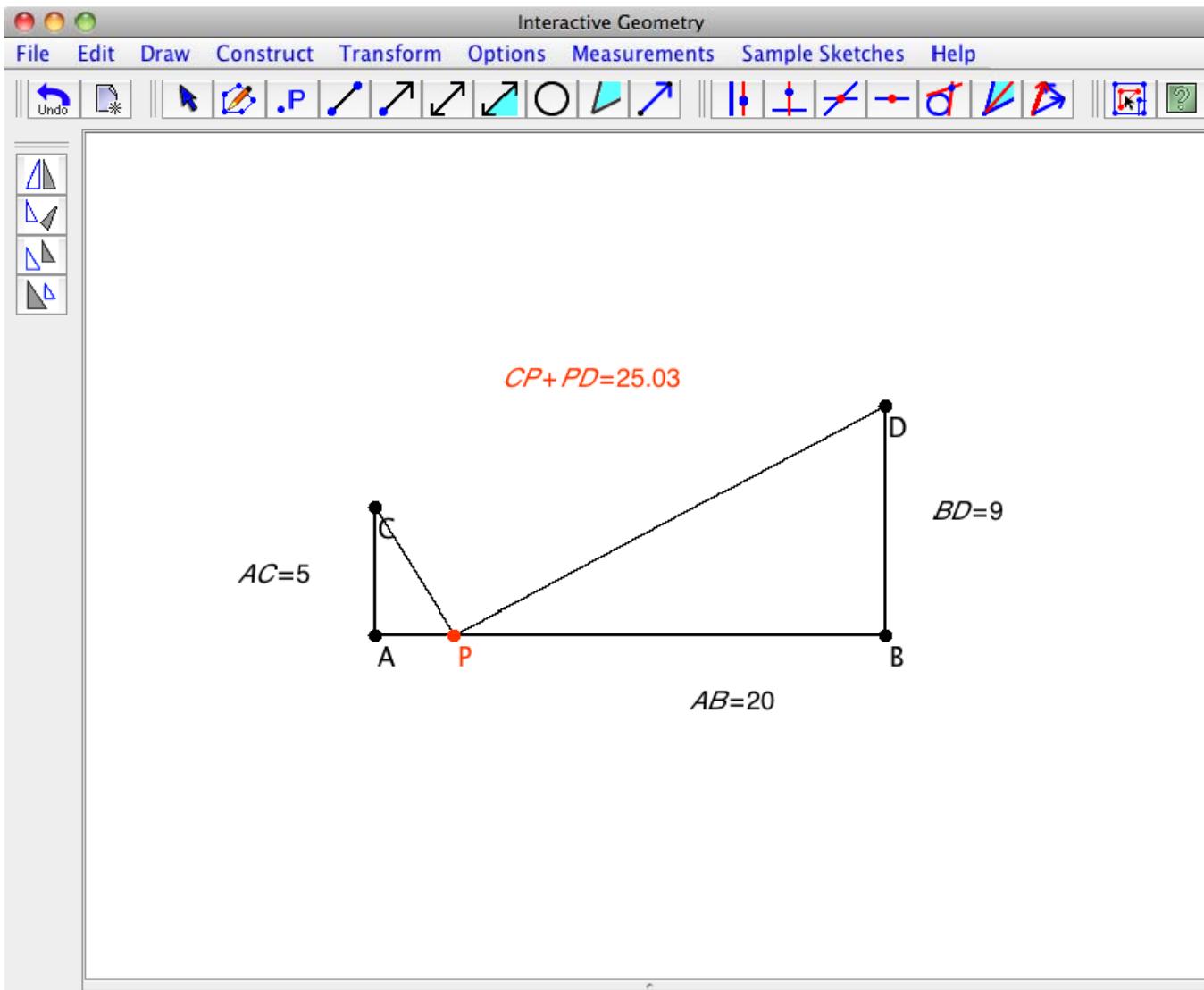
What are important considerations in locating the refinery?

What is your best estimate for the location of the refinery? How did you decide on that location?

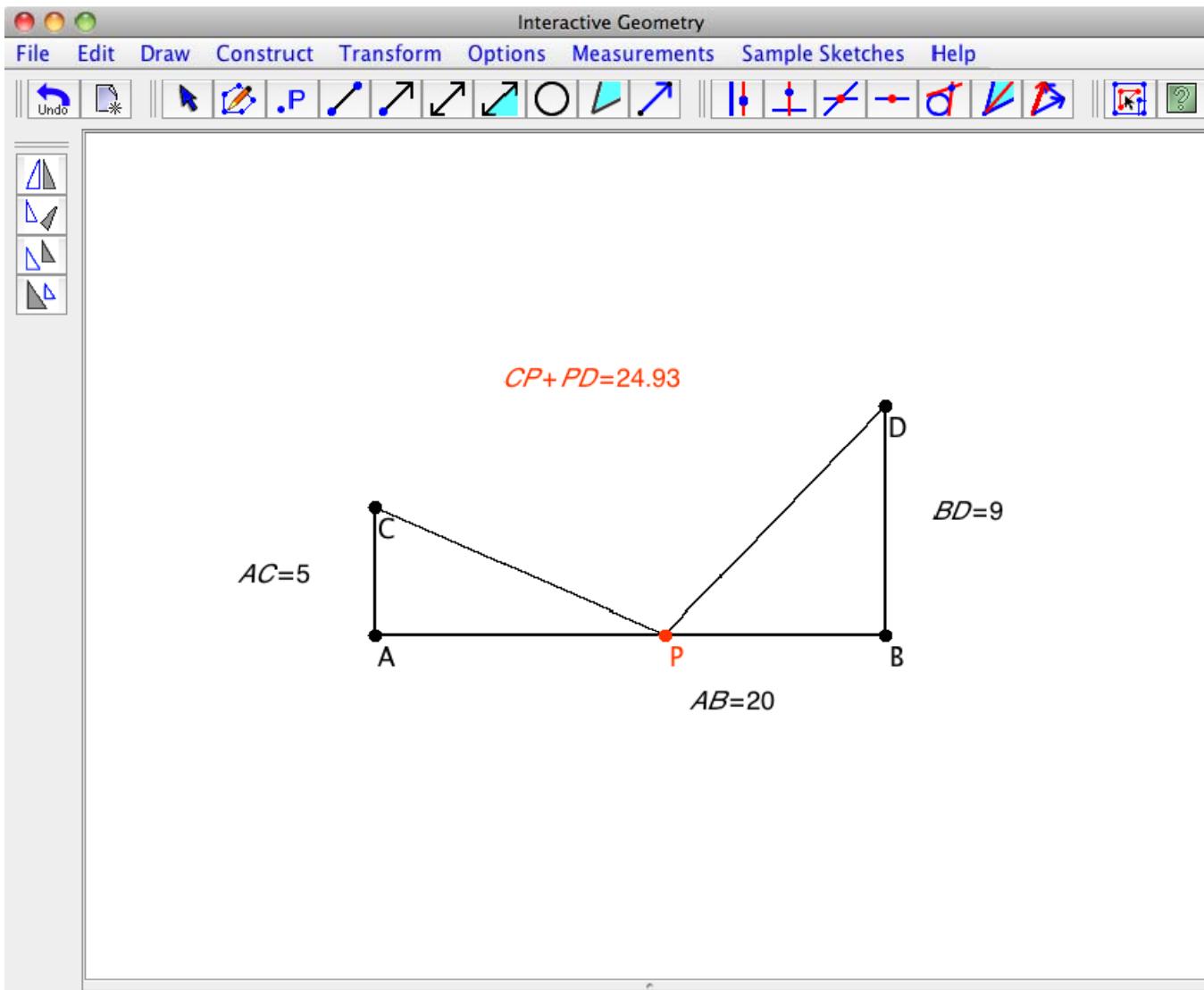




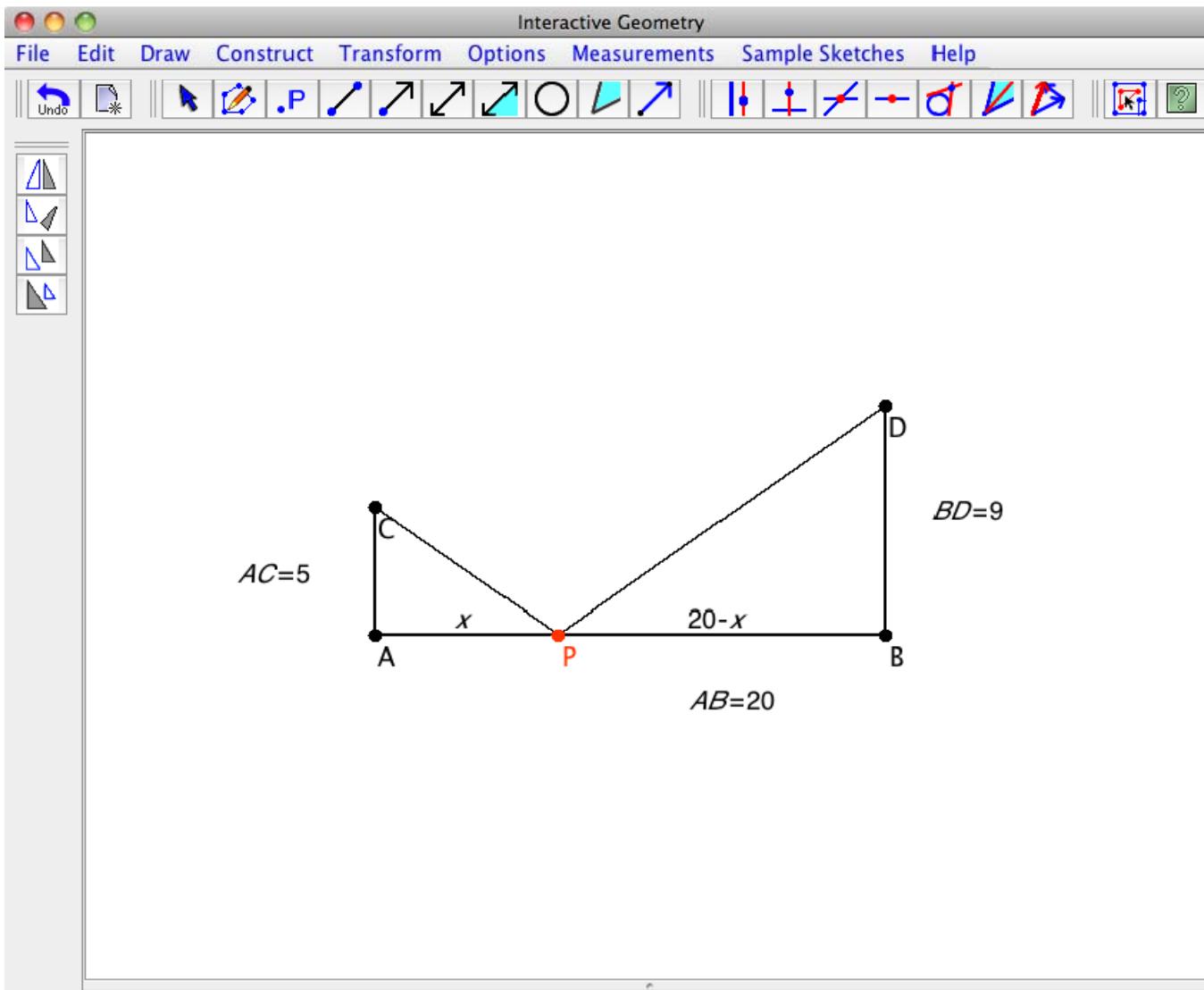
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Spreadsheet

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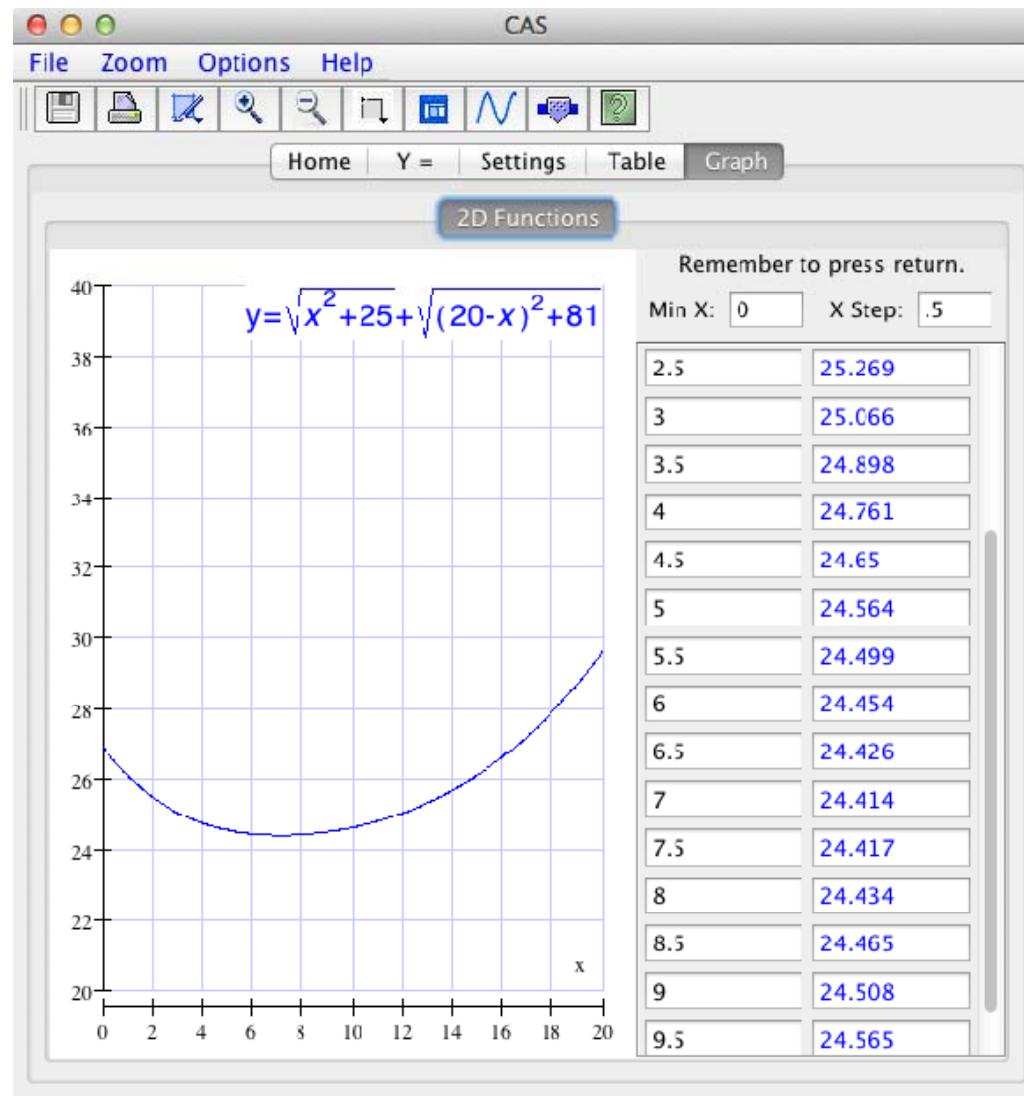
Cell D13: =B13+C13

Data Sheet 1

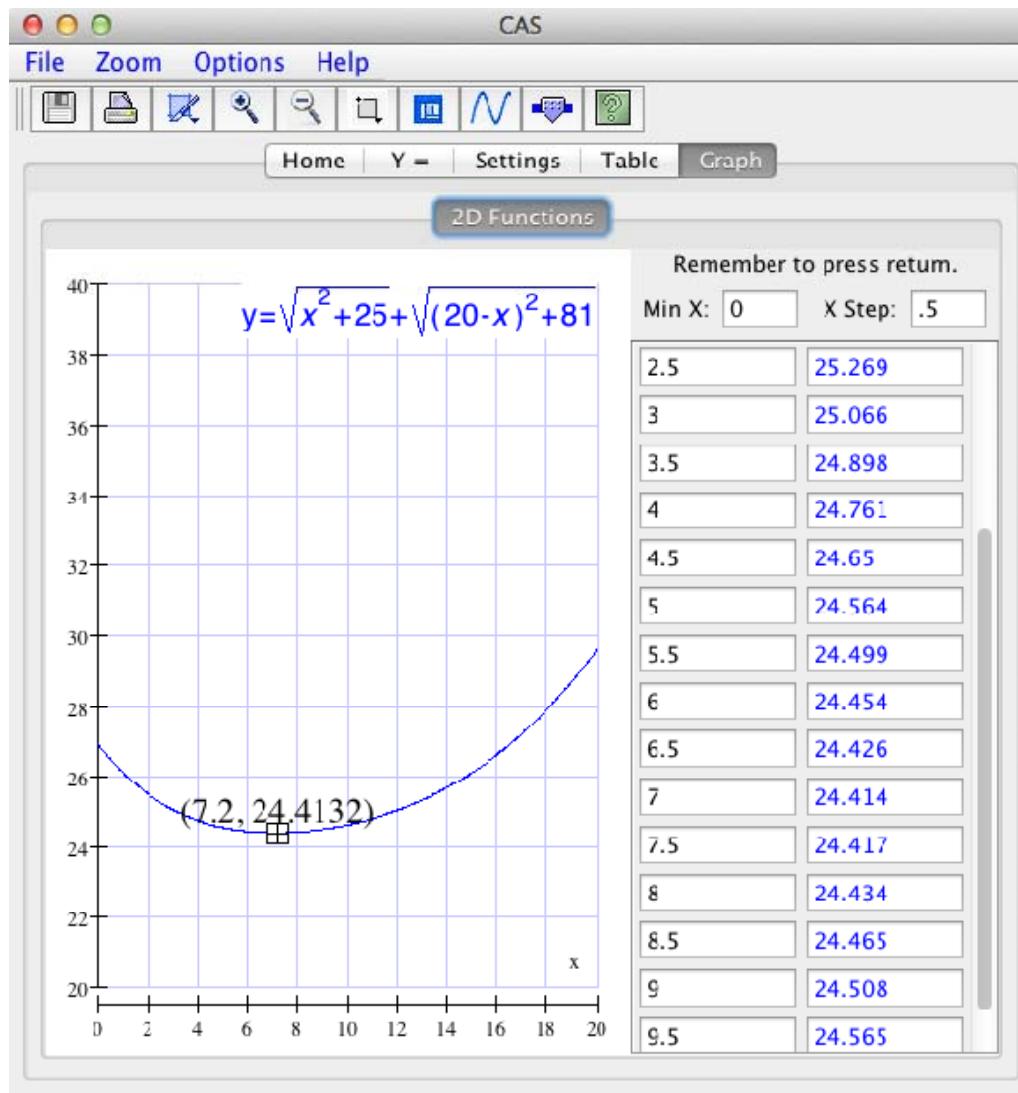
	A	B: CP	C: PD	D: CP+PD	E	F	G
1	x	$\text{sqrt}(A1^2 + 5^2)$	$\text{sqrt}((20-A1)^2 + 9^2)$	$B1 + C1$			
2	6.0	7.81025	16.64332	24.45357			
3	6.1	7.88733	16.55929	24.44662			
4	6.2	7.96492	16.47544	24.44036			
5	6.3	8.04301	16.39177	24.43478			
6	6.4	8.12158	16.30828	24.42986			
7	6.5	8.20061	16.22498	24.42559			
8	6.6	8.28010	16.14187	24.42197			
9	6.7	8.36002	16.05895	24.41898			
10	6.8	8.44038	15.97623	24.41661			
11	6.9	8.52115	15.89371	24.41486			
12	7.0	8.60233	15.81139	24.41371			
13	7.1	8.68389	15.72927	24.41317			
14	7.2	8.76584	15.64736	24.41321			
15	7.3	8.84816	15.56567	24.41383			
16	7.4	8.93085	15.48419	24.41503			
17	7.5	9.01388	15.40292	24.41680			
18	7.6	9.09725	15.32188	24.41913			
19	7.7	9.18096	15.24106	24.42202			
20	7.8	9.26499	15.16047	24.42546			
21	7.9	9.34933	15.08012	24.42945			
22	8.0	9.43398	15.00000	24.43398			
23							



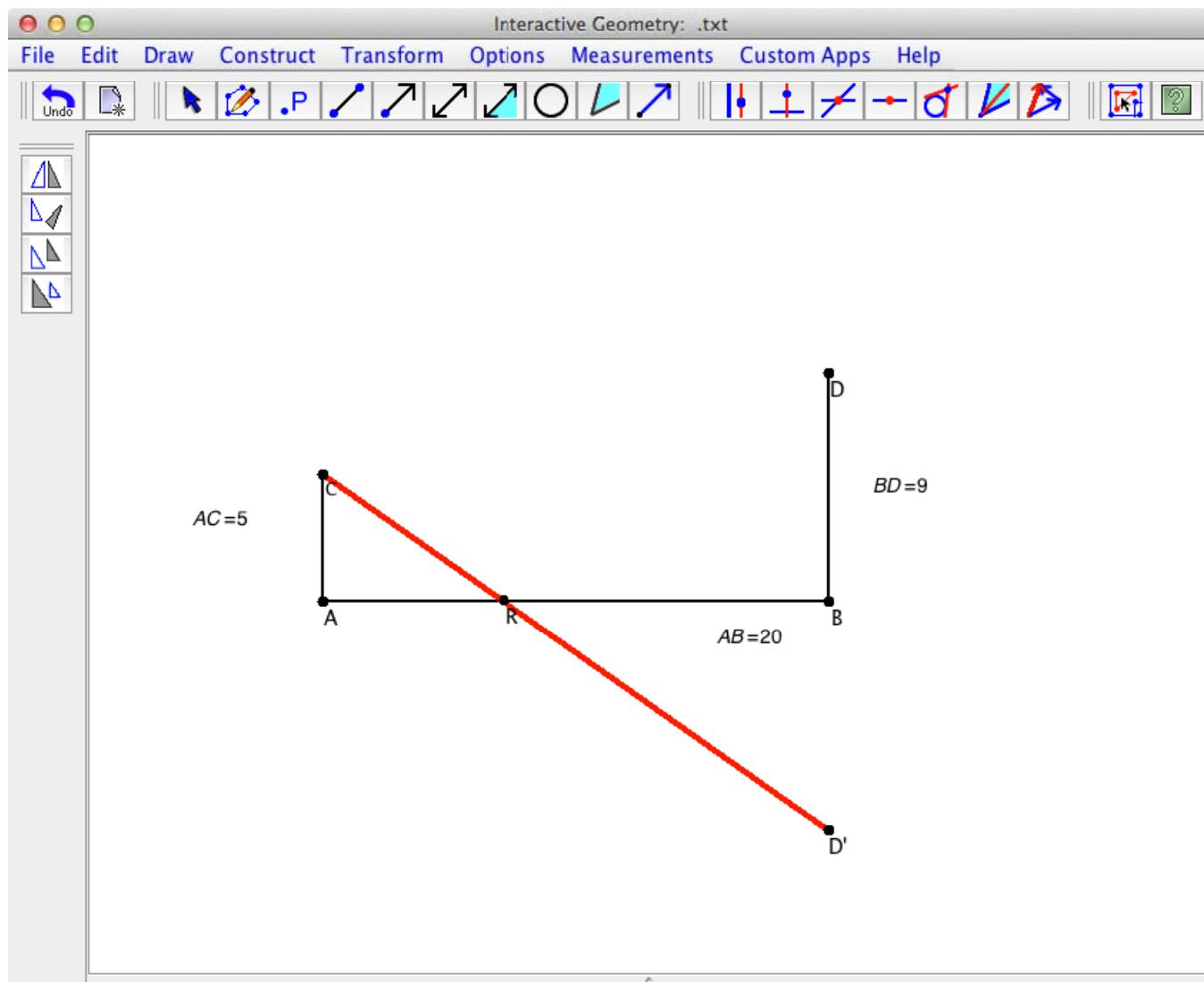
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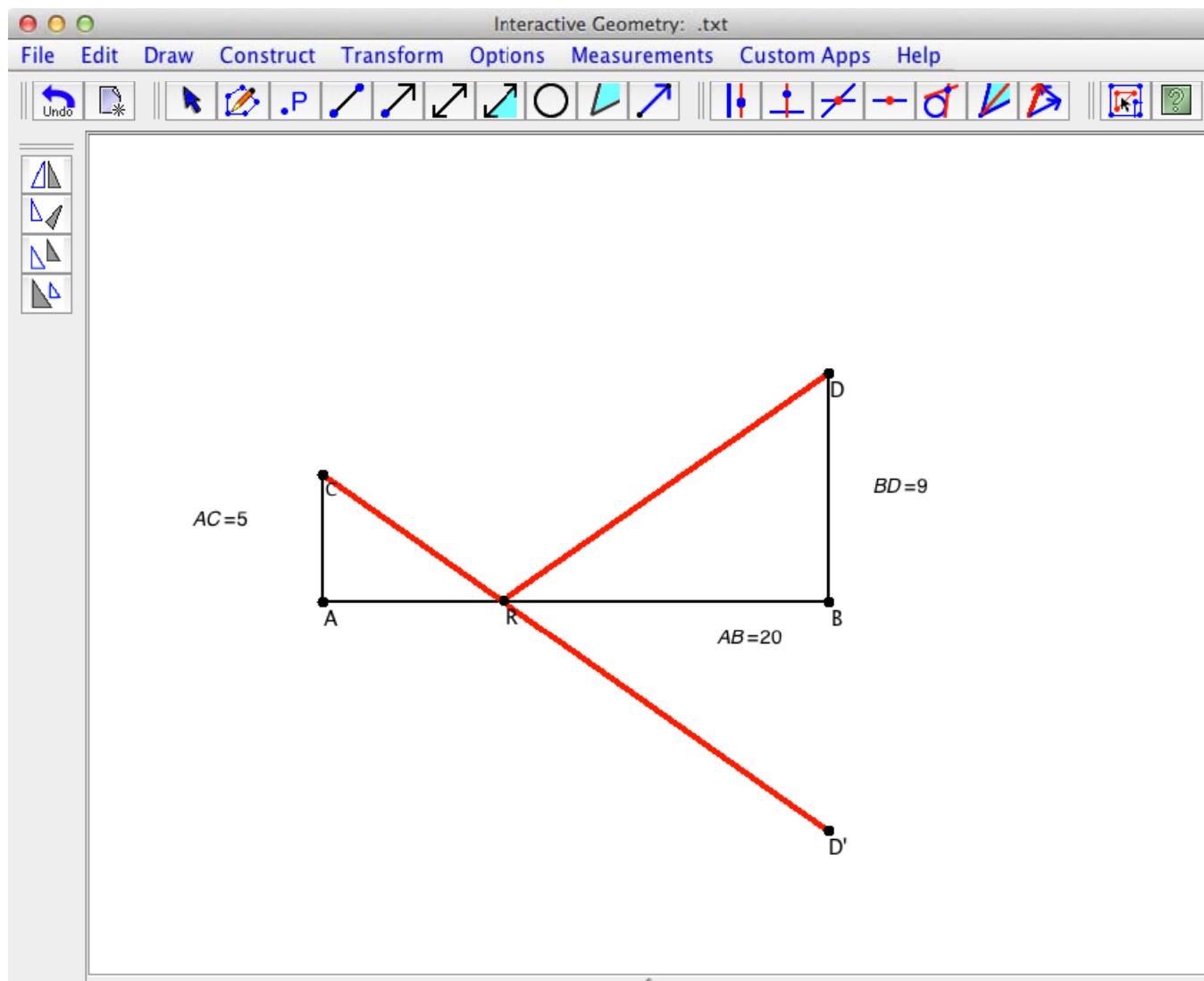
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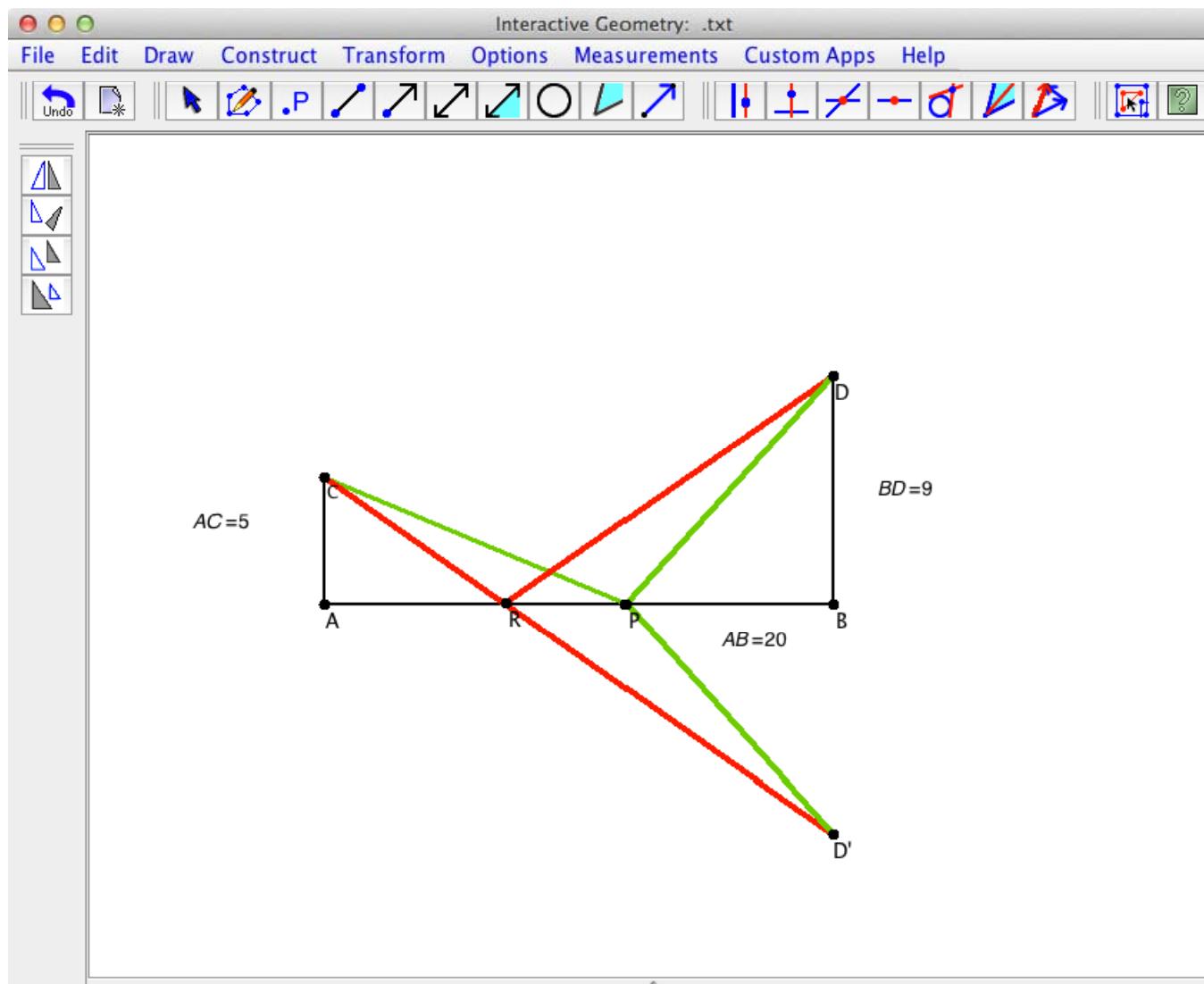
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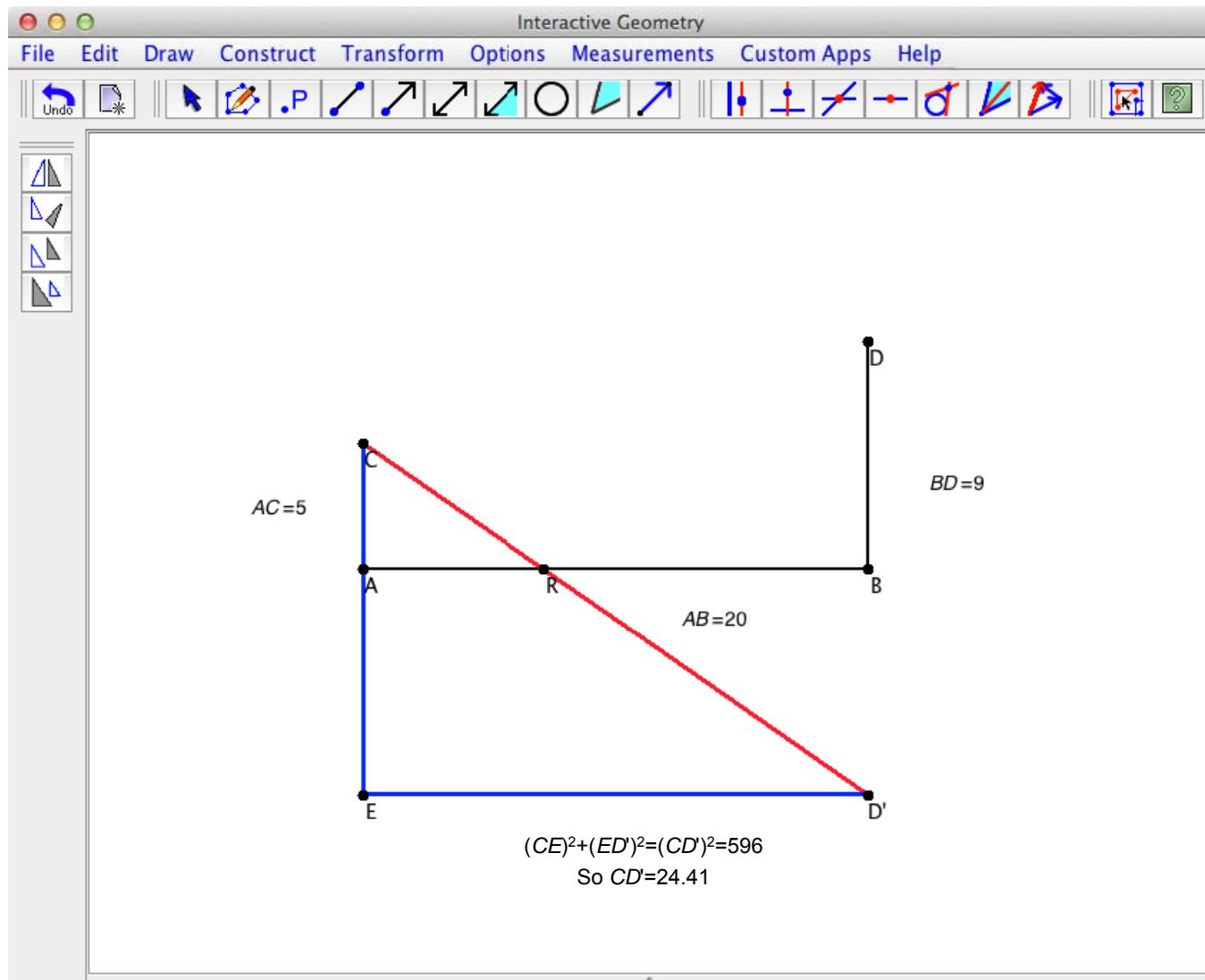


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VIDEO GAME SYSTEM PRODUCTION PROFIT

The manager of TK Electronics must plan for production of two video game systems, a standard model (SM) and a deluxe model (DM).

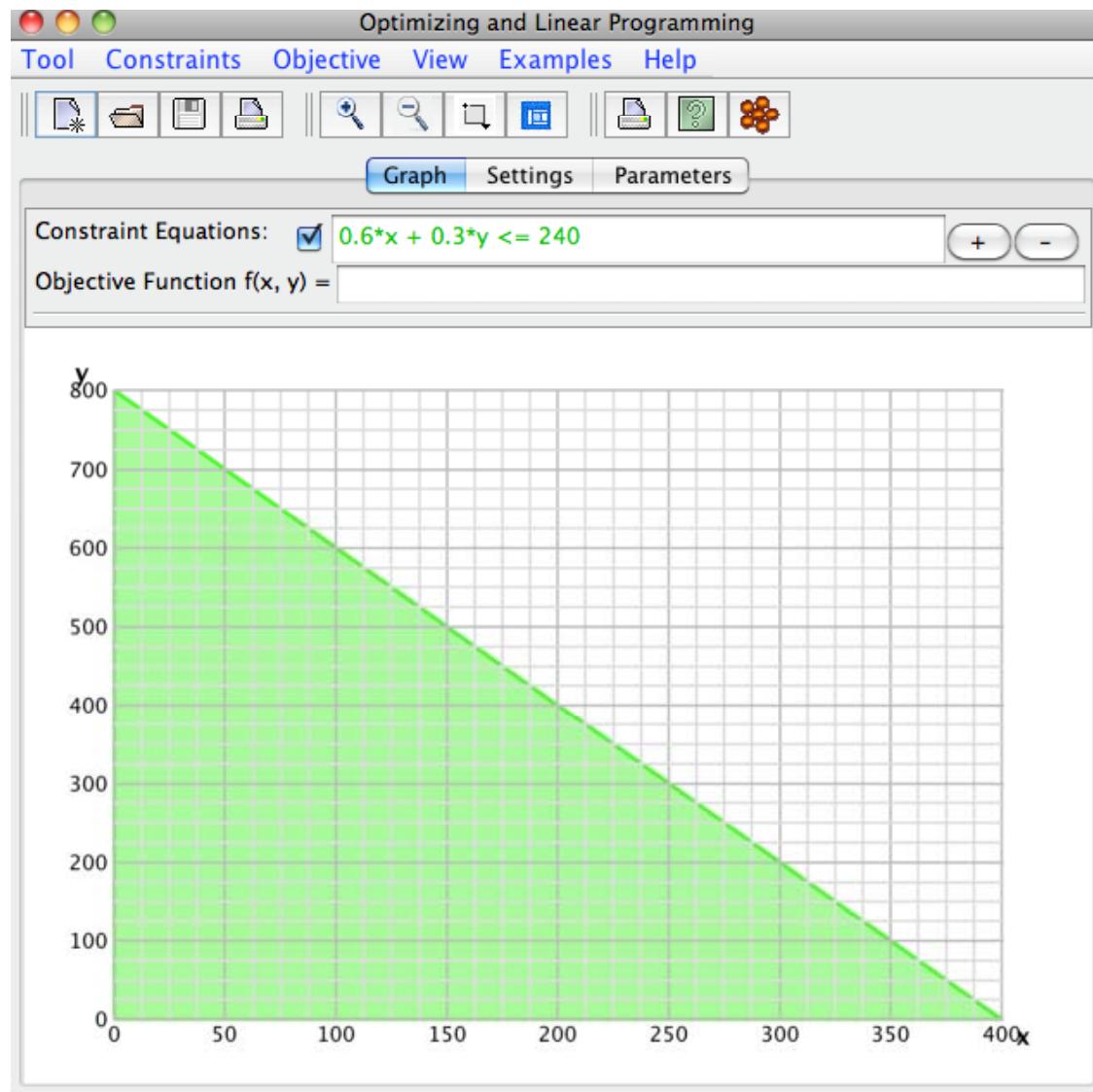
Given the following production limits for assembly time, testing time, and packaging time, how should the manager plan production to maximize profit for his company?



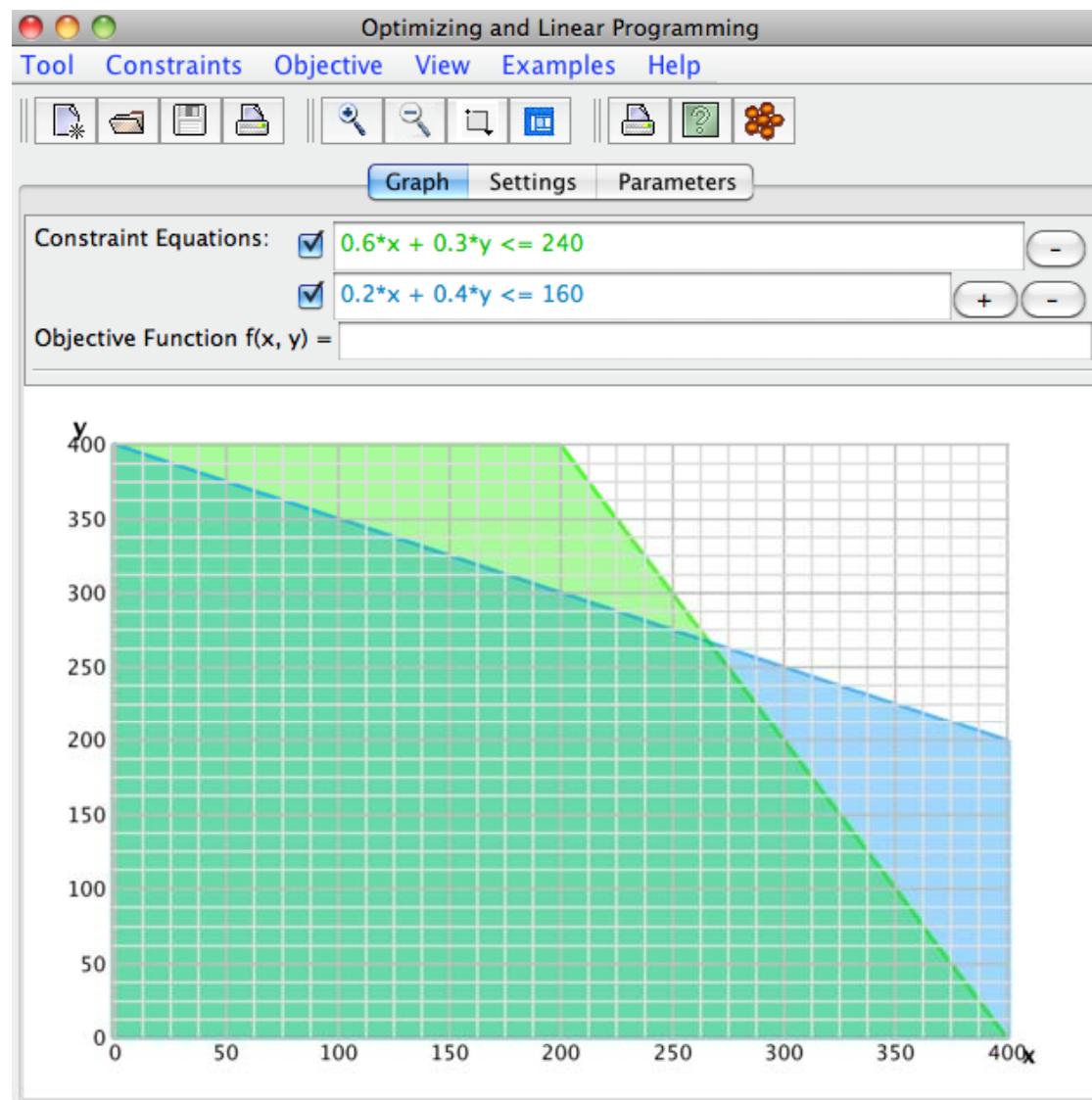
Production Conditions

- Assembly of each SM game system takes 0.6 hours of technician time and assembly of each AM game system takes 0.3 hours of technician time. The plant limits technician time to at most 240 hours per day.
- Testing for each SM system takes 0.2 hours and testing of each AM system takes 0.4 hours. The plant can apply at most 160 hours of technician time each day for testing.
- Packaging time is the same for each model. The packaging department of the plant can handle at most 500 game systems per day.
- The company makes a profit of \$50 on each SM model and \$75 on each AM model.

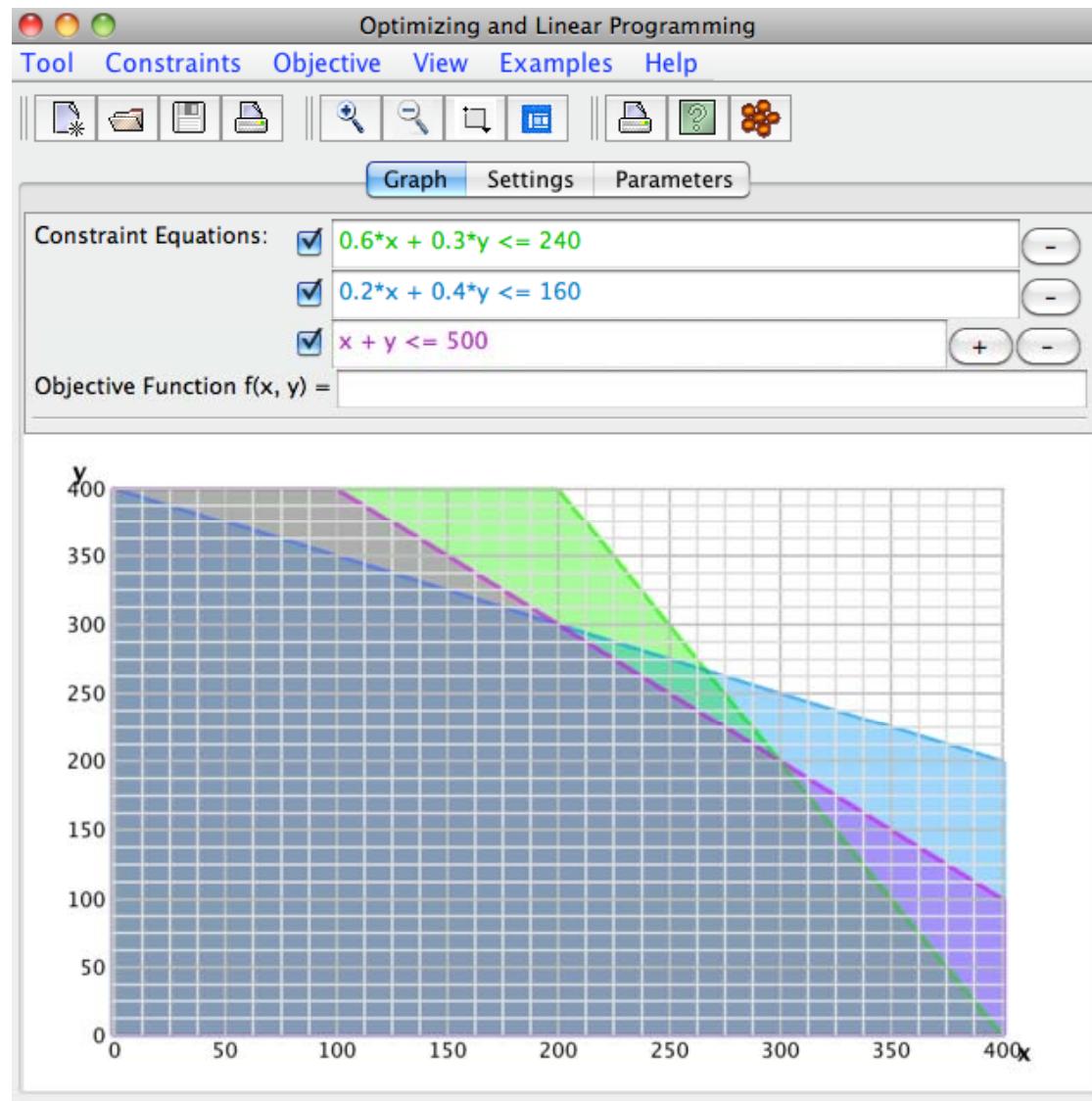




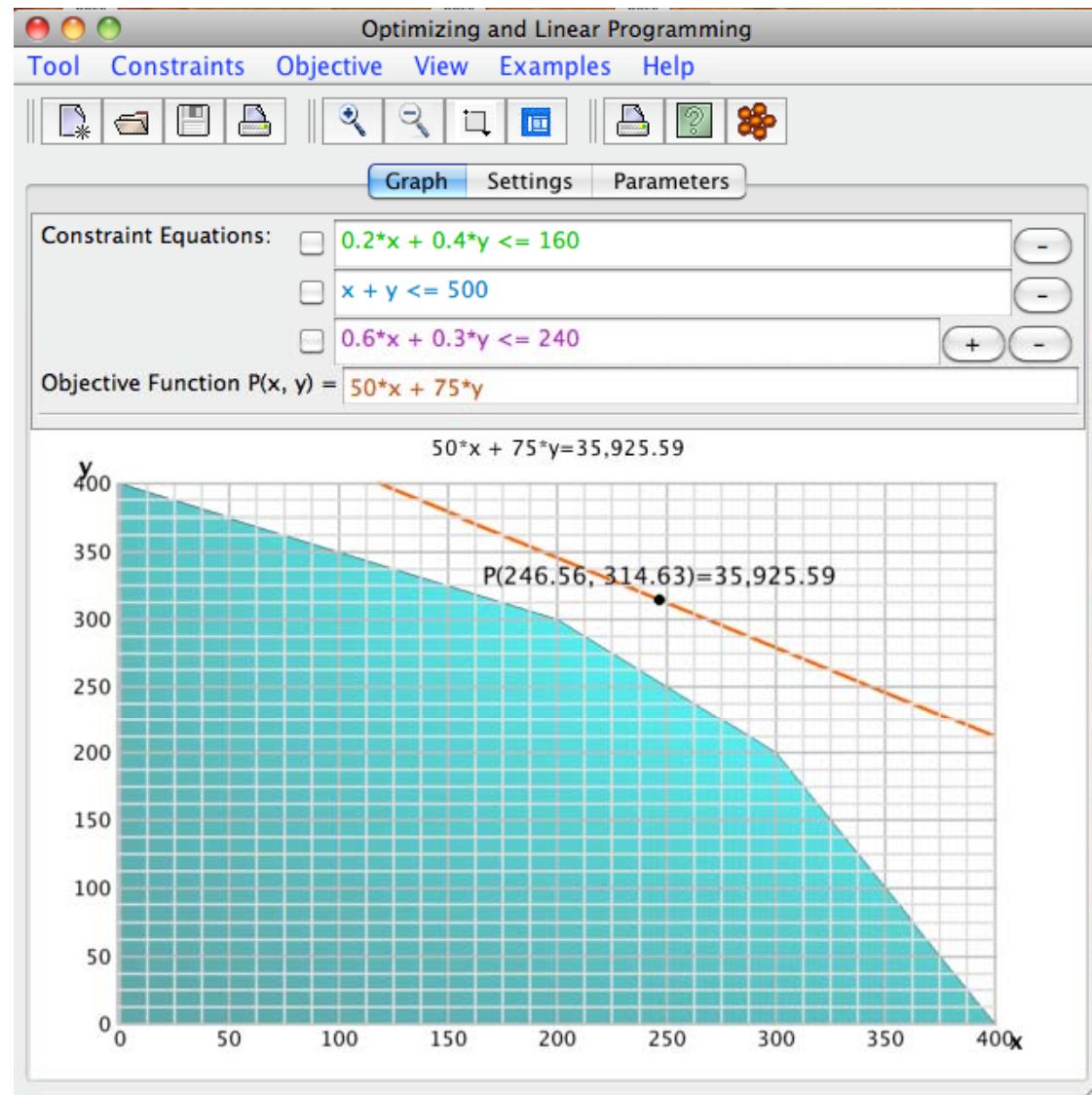
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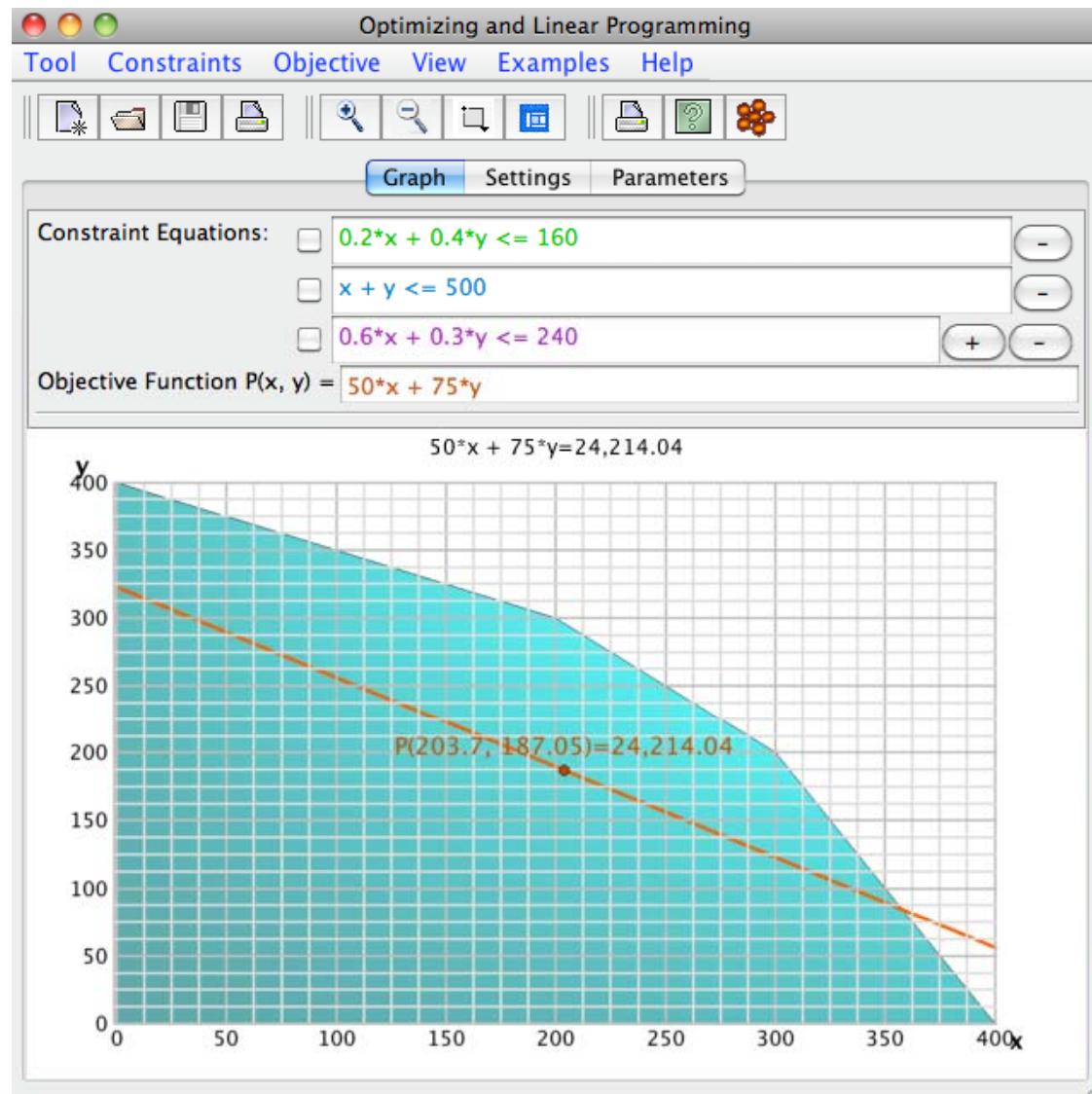
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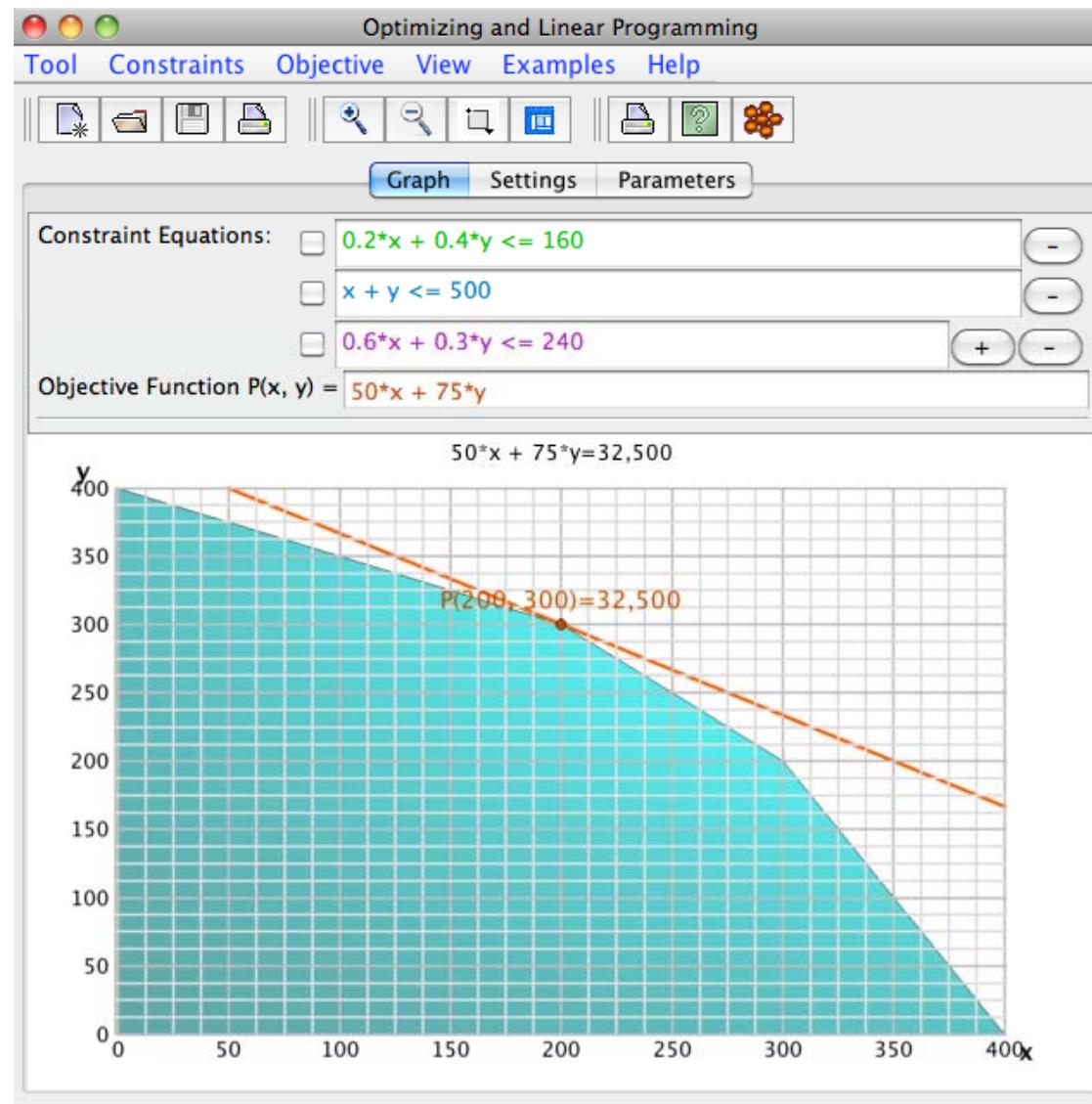
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EYE-HAND COORDINATION

How many pennies
can you stack using
your dominant hand?

How many pennies
can you stack using
your nondominant
hand?

What is a natural
hypothesis to test?



A class of 54 students at Traverse City West High School conducted an experiment. The students were randomly assigned to use their dominant (nondominant) hand to stack the pennies.

The mean for the dominant hand group was approximately 32.89 pennies. The mean for the nondominant hand group was approximately 27.33 pennies.

So, *dominant mean – nondominant mean* \approx 5.56 pennies.



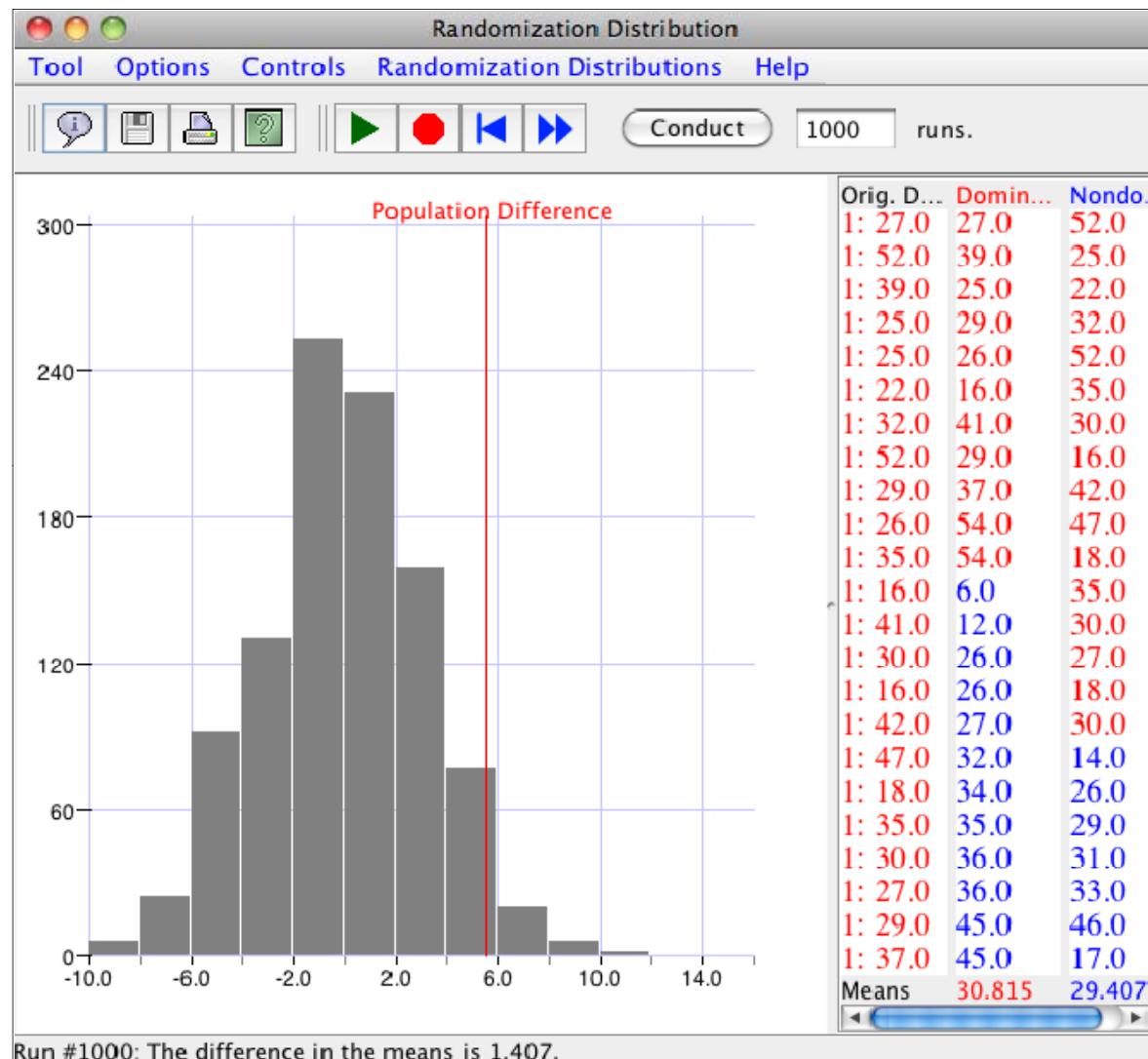
Randomization Tests

- Suppose that each subject will respond the same way no matter which treatment he or she gets. Call this the *null hypothesis*.
- Randomly divide the available subjects into the two treatment groups, give the treatments, and record the responses.
- Generate a *randomization distribution* that shows the difference in the mean response from many different possible randomizations of the subjects to the treatments, still assuming that the null hypothesis is true.
- Decide if the difference from the actual experiment would be extreme (a rare event) if the null hypothesis is true.
- If not, you cannot reject the null hypothesis. If so, you can reject the null hypothesis and conclude that the treatments did make a difference in the mean response.



If statistics software is not available, write the numbers of pennies stacked by the members of your class on identical small slips of paper. Mix them up well. Draw out half of them to represent the students who used their dominant hand. Compute the mean number of pennies stacked. Compute the mean for the remaining slips of paper (that represent those using their nondominant hand). Subtract *dominant mean – nondominant mean*. Repeat until you have 100 differences.



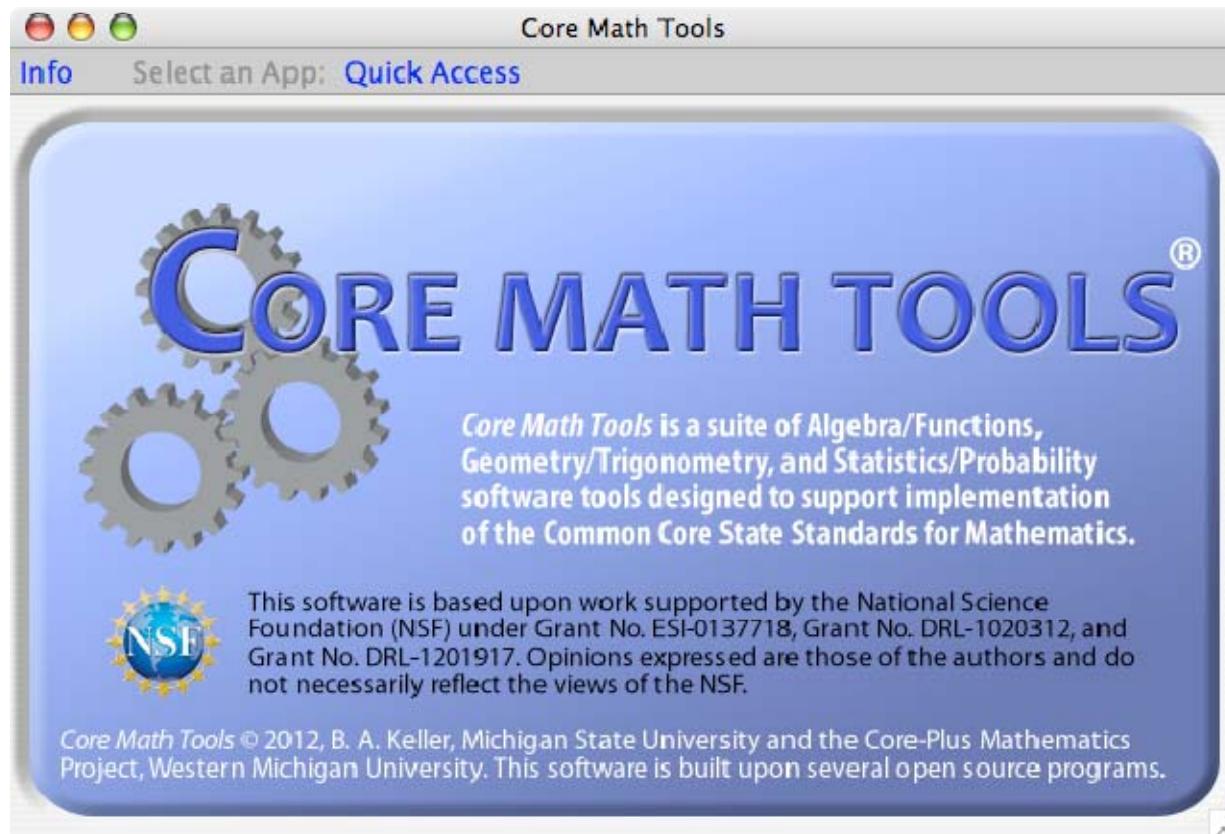


Responses will vary depending on your class results. If the randomization distribution shows a probability of 0.05 or less of getting a difference as extreme or even more extreme as the actual difference from your class, then you should conclude that the hand used made a difference.

Based on the displayed randomization distribution, about 71 of the 1,000 differences were above 5.56 or below –5.56. So, the probability of getting a difference as extreme as 5.56 just by chance is about 0.071.

For this class, use of dominant hand appeared to have some influence on the number of pennies stacked, but that influence was not significant.





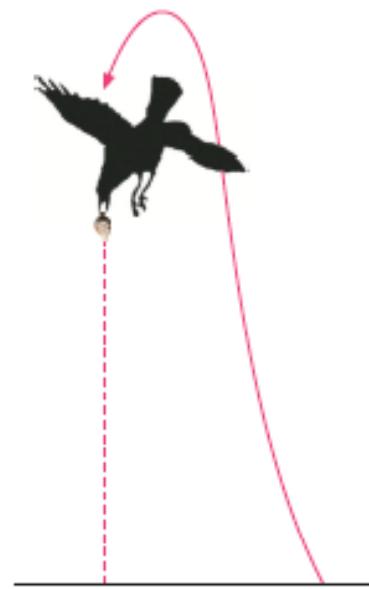
www.nctm.org/coremathtools



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DROPPING WHELKS

One of the most intriguing natural examples of dropping with intent to break is exhibited by sea gulls and crows who feed on mollusks that have shells, like snails and clams. Biologists have observed a species of crows that pick up *whelks*, lift them into the air, and drop them on rocks to break open the shells.



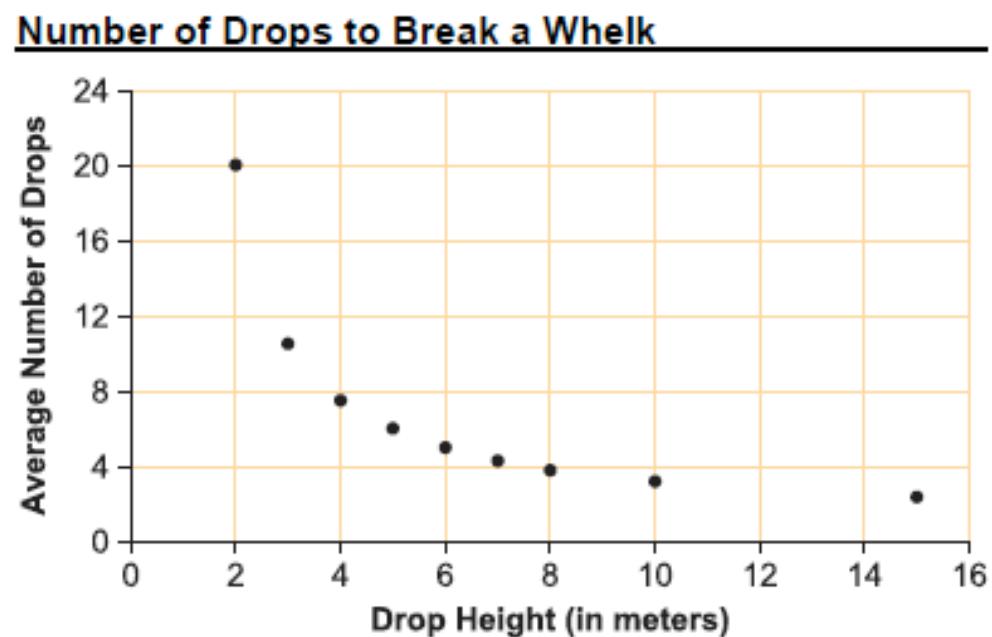
What has especially intrigued biologists who observe the whelk-dropping behavior of northwestern crows is the uncanny way that they seem to rise consistently to a height of about 5 meters before dropping the shells onto the ground.

- a. What considerations might influence the crows' choice of 5 meters as an optimal drop height?
- b. What relationship would you expect between the number of drops it takes to break a whelk shell and the height from which the shell is dropped?
- c. What kind of experiment could you do to simulate a model relating *number of drops* before breaking to *height* of the drops?



Canadian biologist Reto Zach performed an experiment in which he dropped whelks from many different heights and recorded the number of drops it took to break the whelk shell in each case. His data gave a pattern like that in the table and graph.

Drop Height (in meters)	Average Number of Drops
2	20
3	10.5
4	7.5
5	6
6	5
7	4.3
8	3.8
10	3.2
15	2.4



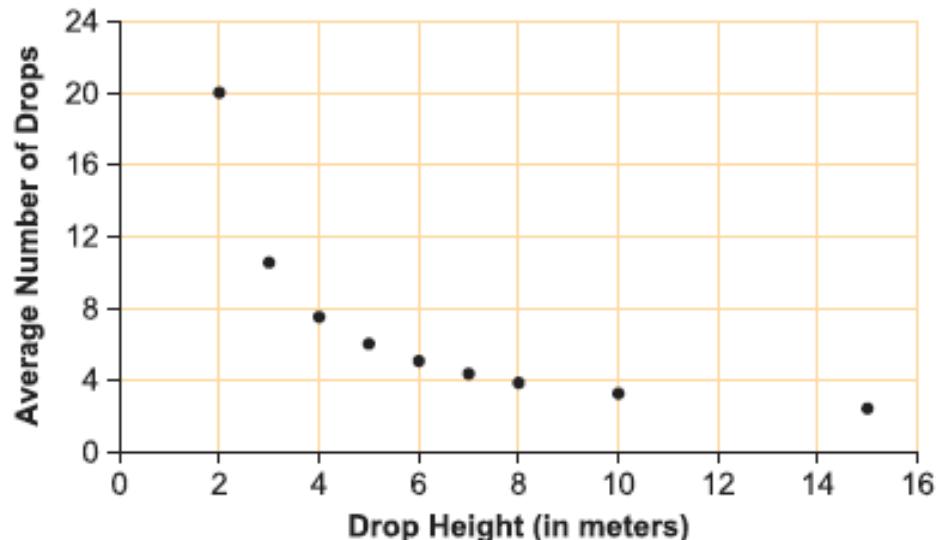
Study the pattern of data, the shape of the graph, and the problem situation being modeled. What function family might provide a good model for the relationship between number of drops and drop height?

Inverse variation

How might you customize the “parent” of the function family to obtain a better-fitting model?

Graph of $y = \frac{1}{x}$ can be translated upward to produce a horizontal asymptote. It has y -axis as a vertical asymptote.

Number of Drops to Break a Whelk



One proposed model for the relationship between number of drops and drop height was the function $N(h) = 1 + \frac{200}{10h - 9}$

Test the plausibility of this model by analyzing the rule itself, without using a graph of the rule.

a. How will the values of $N(h)$ change as the value of h increases?

Approaches 1 as a lower limit

b. How will the values of $N(h)$ change as the value of h decreases toward 1?

Approaches 201

c. What is the lower bound for the value of $N(h)$ when $h > \frac{9}{10}$

d. How do you think this model was developed?



One reason that the crows might not choose to lift the whelks very high is the work required by that task. In physics, work done in moving an object is calculated by combining the force required and the distance the object moves. Data in the following table show the amount of work (in joules) that might be required to lift a typical whelk to various heights.

Height (in meters)	2	4	6	8	10	12	14	16
Work (in joules per drop)	6	12	18	24	30	36	42	48

- a. What function $W(h)$ shows how work required for lifting a whelk depends on height?

$$W(h) = 3h$$

- b. Continue using the proposed model for the relationships between number of drops and drop height as before. Suppose that a crow chose to consistently drop whelks from a height of 8 meters.

- i. How many drops from that height would be required to break a whelk?

$$N(8) = 3.8$$

- ii. How much total work would be required to drop a whelk until it breaks?

$$W(8) = 24 \quad \text{So total work is } (24)(3.8) = 91.2 \text{ joules}$$



- c. Suppose that the crow consistently dropped its whelks from a height of 4 meters.
- i. How many drops from that height would be required to break a whelk?

$$N(4) = 7.45 \text{ drops}$$

- ii. How much total work would be required to drop a whelk until it breaks?

$$W(4) = 12 \quad \text{So total work is } (12)(7.45) = 89.4 \text{ joules}$$

- d. For crows that choose some consistent height h for their whelk drops:
- i. what function $WB(h)$ shows how to calculate the average work required to break a whelk as a function of drop height?

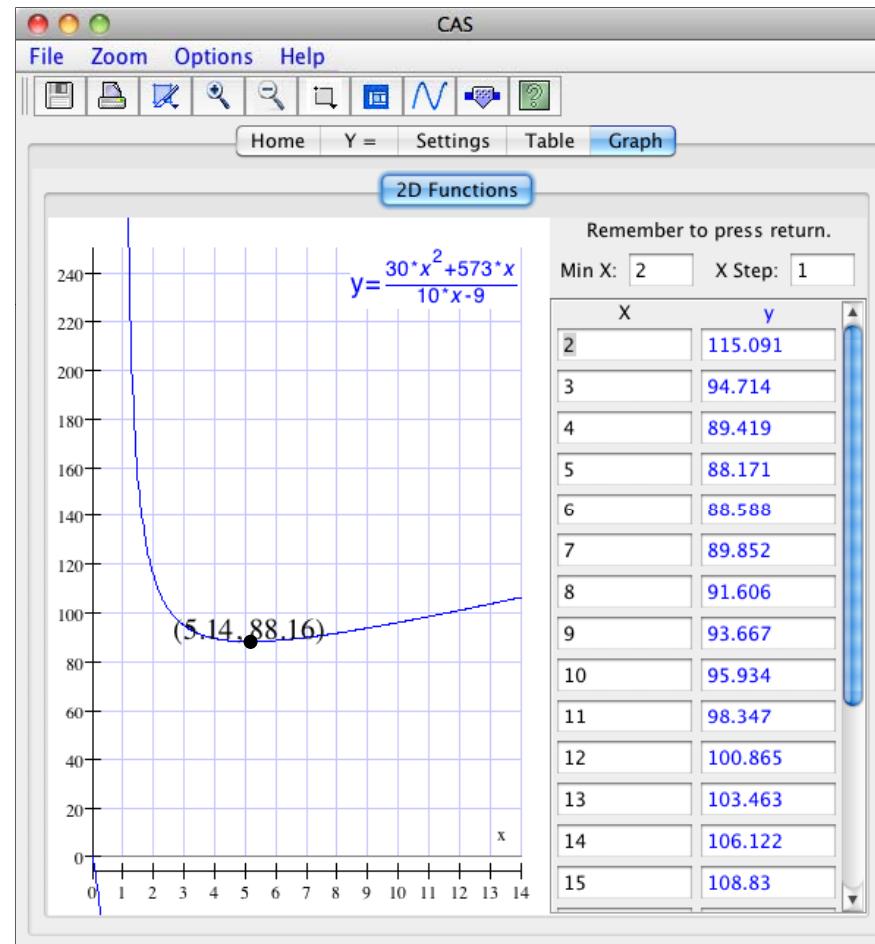
$$WB(h) = N(h) \cdot W(h) = \left(1 + \frac{200}{10h - 9}\right)(3h)$$

- ii. write the rule for $WB(h)$ as a single rational expression.

$$WB(h) = \frac{30h^2 + 573h}{10h - 9}$$



- e. Estimate the drop height which, if used consistently, will break whelk shells with the least total work by the crows.



Comments or Questions?



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