Promoting Algebraic Reasoning: 
*The Role of Mathematical Tasks*

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Algebra Readiness for Every Student
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Promoting algebraic reasoning in the classroom involves incorporating conjecture, argumentation, and generalization in purposeful ways so that students consider arguments as ways to build reliable knowledge. It requires respecting and encouraging these activities as standard daily practice, not as occasional “enrichment”...

Critical to algebraic thinking is the capacity to recognize patterns and organize data to represent situations in which input is related to output by well-defined functional rules.

Overview

- Discuss tasks that have the potential to support algebraic reasoning
- Analyze and discuss the ways in which teachers support algebraic reasoning
- Consider how use of the *Effective Mathematics Teaching Practices* in *Principles to Action* (NCTM, 2014) can help support algebraic reasoning
Comparing Two Pattern Tasks

- Compare two pattern tasks and consider how they are the same and how they are different
- Consider the opportunities each task provides to engage in Algebraic Reasoning
Comparing Two Pattern Tasks

**Hexagon** (blue)
Trains 1, 2, 3 and 4 are the first 4 trains in the hexagon pattern. The first train in this pattern consists of one regular hexagon. For each subsequent train, one additional hexagon is added.

1. Compute the perimeter for each of the first four trains;
2. Draw the fifth train and compute the perimeter of the train;
3. Determine the perimeter of the 25th train without constructing it;
4. Write a description that could be used to compute the perimeter of any train in the pattern and explain why it works; and
5. Determine which train has a perimeter of 110.

**Patterns** (green)
The table of values below describes the perimeter of each figure in the pattern of blue tiles. The perimeter $P$ is a function of the number of tiles $t$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

a. Choose a rule to describe the function in the table.
   A. $P = t + 3$  
   B. $P = 4t$
   C. $P = 2t + 2$  
   D. $P = 6t - 2$

b. How many tiles are in the figure if the perimeter is 20?

c. Graph the function.
### Comparing Two Tasks

#### Same
- The tasks involve trains of regular polygons
- When the train number increases by 1 an additional polygon is added to the train
- The number of polygons is the same as the train number
- The tasks ask you to find the number of tiles in a train given the perimeter
- A knowledge of perimeter is needed to solve the tasks

#### Different
- Multiple ways to enter Hexagon – make table, build trains, inspect diagram
- Multiple ways to solve Hexagon - table, graph, different equations that model the physical arrangement of tiles
- In the Patterns task, the table is done, the equations are provided, very little thinking is needed. Part a is a matching task and b is a plug and chug task. While you are asked to graph in part c, this is nothing more than a plotting points exercise.

## Opportunities for Algebraic Reasoning

### Hexagon Task

- **Recognize Patterns** – for each train, another hexagon is added; the number of hexagons in a train is the same as the train number; each train has a perimeter that is 4 more than the previous train.

- **Relate Input to Output** – describing how the perimeter of a train (output) is related to the train number (input).

- **Creating a Generalization** – describing verbally or algebraically how the input and output are related.

- **Explaining Why and How It Works** – creating an argument that defends your reasoning and shows that it is true for all cases.

### Pattern Task

There aren’t any!!!!!
### The Task Analysis Guide

<table>
<thead>
<tr>
<th>Lower-Level Demands</th>
<th>Higher-Level Demands</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Memorization</strong></td>
<td><strong>Procedures With Connections</strong></td>
</tr>
<tr>
<td>• involve either reproducing previously learned facts, rules, formulae or definitions OR committing facts, rules, formulae or definitions to memory.</td>
<td>• focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.</td>
</tr>
<tr>
<td>• cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.</td>
<td>• suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.</td>
</tr>
<tr>
<td>• are not ambiguous. Such tasks involve exact reproduction of previously-seen material and what is to be reproduced is clearly and directly stated.</td>
<td>• usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning.</td>
</tr>
<tr>
<td>• have no connection to the concepts or meaning that underlie the facts, rules, formulae or definitions being learned or reproduced.</td>
<td>• require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.</td>
</tr>
</tbody>
</table>

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<tr>
<th>Procedures Without Connections</th>
<th><strong>Doing Mathematics</strong></th>
</tr>
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<tbody>
<tr>
<td>• are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task.</td>
<td>• require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example).</td>
</tr>
<tr>
<td>• require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it.</td>
<td>• require students to explore and understand the nature of mathematical concepts, processes, or relationships.</td>
</tr>
<tr>
<td>• have no connection to the concepts or meaning that underlie the procedure being used.</td>
<td>• demand self-monitoring or self-regulation of one's own cognitive processes.</td>
</tr>
<tr>
<td>• are focused on producing correct answers rather than developing mathematical understanding.</td>
<td>• require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.</td>
</tr>
<tr>
<td>• require no explanations or explanations that focuses solely on describing the procedure that was used.</td>
<td>• require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.</td>
</tr>
<tr>
<td>• require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.</td>
<td></td>
</tr>
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</table>
Implement Tasks that Promote Reasoning and Problem Solving

Mathematical tasks should:

• Provide opportunities for students to engage in exploration or encourage students to use procedures in ways that are connected to concepts and understanding;

• Build on students’ current understanding; and

• Have multiple entry points.

There is no decision that teachers make that has a greater impact on students’ opportunities to learn and on their perceptions about what mathematics is than the selection or creation of the tasks with which the teacher engages students in studying mathematics.

Comparing Two Versions of a Task: Fun Tees (green)

**Fun Tees– Version 1**

Fun Tees is offering a 30% discount on all merchandise. Find the amount of discount on a T-shirt that was originally priced at $16.00.

**Fun Tees – Version 2**

Fun Tees is offering a 30% discount on all merchandise.

- Find the amount of discount on a T-shirt that was originally priced at $16.00.
- Suppose the T-shirt was originally priced at $17, $18, 19, 20, or $50. Describe the amount of discount on t-shirts at each price.
- Write a number sentence that describes how much discount you will receive on any T-shirt that is offered at a 30% discount.

How are these two tasks the same and how are they different?
Comparing Two Versions of a Task: Fun Tees (green)

Same

• Both versions of the task ask you to find the amount of discount on a T-shirt that is regularly priced at $16.00 and is marked 30% off.

Different

• V1 of the task requires the use of a learned rule or procedure. It usually appears in a text after you have seen worked examples.
• V2 requires going beyond performing the calculation to asking students to find and describe a pattern
  ○ Suppose the T-shirt was originally priced at $17, $18, 19, 20, or $50. Describe the amount of discount on t-shirts at each price.
• V2 requires generalizing the pattern to find the amount of discount on any T-shirt
  ○ Write a number sentence that describes how much discount you will receive on any T-shirt that is offered at a 30% discount.
Consider Two Additional Tasks

**S-Pattern**

1. What patterns do you notice in the set of figures?
2. Sketch the next two figures in the sequence.
3. Describe a figure in the sequence that is larger than the 20th figure without drawing it.
4. Determine an equation for the total number of tiles in any figure in the sequence. Explain your equation and show how it relates to the visual diagram of the figures.
5. If you knew that a figure had 9802 tiles in it, how could you determine the figure number? Explain.
6. Is there a linear relationship between the figure number and the total number of tiles? Why or why not?

**Supreme Court Handshake**

When the nine justices of the Supreme Court Justices meet each day, each shakes the hand of every other justice, to show harmony of aims, if not views.

1. If each justice shakes hands exactly once with each of the other justices, how many handshakes take place? Explain.
2. How can you determine the number of handshakes for a group too large to model? Explain.

http://illuminations.nctm.org/lesson.aspx?id=2112
So What Have We Established So Far?

In order to engage students in algebraic reasoning, you need:

1. A **high-level (cognitively demanding) mathematical task** that provides students with opportunities to:
   - Look for patterns
   - Make generalizations
   - Explain reasoning

2. To make sure that all students can enter the task and that there is sufficient challenge in the task for all students (use the task to differentiate as needed)
If we want students to develop the capacity to think, reason, and problem solve then we need to start with high-level, cognitively complex tasks.

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Role of the Teacher

Teachers must decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them and thus eliminating the challenge.

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• What evidence can you find of students engaging in algebraic reasoning?

• What did the Ms. Peterson do to support her students’ opportunity to reason algebraically?
Evidence of Algebraic Reasoning

- **Students recognized patterns**
  - Each time a hex is added the perimeter increases by 4 and the two end sides get added on (G1 & G3)
  - The first and last hex in a train each contribute 5 sides to the perimeter and the hexagons in the middle contribute 4 (G2)
  - A hexagon has six sides, but they do not all count in the perimeter. You can subtract the vertical sides and add back the two sides on the ends (G5)

- **Students created and explained generalizations**
  - \( P = 4h + 2 \)  (G1)
  - \( P = 4(h-1) + 6 \)  (G4)
  - \( P = 6h - 2h + 2 \)  (G5)
  - \( P = 4(n-2) + 10 \)  (G2 and class)

- **Students related input (number of hexagons) to output (perimeter)**
- **Students organize data (G3)**
How Ms. Peterson Supported Algebraic Reasoning?

1. Establish mathematics goals to focus learning.
2. Implement tasks that promote reasoning and problem solving.
3. Use and connect mathematical representations.
4. Facilitate meaningful mathematical discourse.
5. Pose purposeful questions.
6. Build procedural fluency from conceptual understanding.
7. Support productive struggle in learning mathematics.
8. Elicit and use evidence of student thinking.
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- T goals were grade-level appropriate and consistent with CCSSM
- T used the goals to guide her decision-making throughout the lesson
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T pressed students to make connections between the visual arrangement of tiles and the equations.
T asked students to sketch the graph based on their descriptions.
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**T monitored students as they worked and knew who did what**

**T determined what solutions would be presented and in what order**

**T asked questions to highlight the math she wanted students to learn**
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T asked questions to:
- Determine what students knew
- Highlight the math she wanted them to learn
- Invite participation
- Encourage mathematical exploration
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- T asked questions to help students make progress without taking over the thinking for them
- T encouraged students to build and examine the hexagon trains
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- T gave students a task that would elicit their thinking
- T used students’ thinking as the basis for the small and whole group discussions
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• What evidence can you find of students engaging in algebraic reasoning?

• What did the Ms. Rossman do to support her students’ opportunity to reason algebraically?
Patricia Rossman: Implementing the Hexagon Train Task

- The video is available at

Evidence of Algebraic Reasoning

- **Students recognized patterns**
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2. To make sure that all students can enter the task and that there is sufficient challenge in the task for all students (use the task to differentiate as needed)

3. To support students’ thinking and reasoning without eliminating the challenging aspects of the task by engaging in the effective mathematics teaching practices.
Thank You!

pegs@pitt.edu
Generalizations Relating Train Number and Perimeter

- $P = 4h + 2$
- $P = 4(h-2) + 10$
- $P = 4(h-1) + 6$
- $P = 6h - 2(h-2) - 2$
- $P = 6h - 2h + 2$
Explaining Why You Add 4 Each Time

The perimeter of the first train is 6, second one is 10, the third one is 14, and the fourth one is 18. Every time you add another hex you just add two sides on the top and two on the bottom. If you look at train two, you have four sides on the top, four on the bottom and the two on the ends. If you look at train three you added one more hex which gives you two more sides on the top and the bottom (see arrows). It doesn’t matter how big the train is, because if you add another hex, you only count 4 more sides.
“Algebrafying”

Transforming problems with a single numerical answer to opportunities for pattern building, conjecturing, generalizing, and justifying mathematical facts and relationships.

Entry and Challenge

- Provide students with tools that can give them access to the task (e.g., pattern blocks for hexagon task)

- Provide scaffolding that allows students to get a foothold on the problem (e.g., parts 1 and 2 of the hexagon task)

- Include a question that will provide a challenge for students who are ready for it (e.g., extension question hexagon task)
Starting the Year with Pattern Tasks

...all students can do something mathematical when presented with a geometric pattern. One teacher noted that regardless of your background, you can fly into the task anywhere. You can have the brightest kid in your class and the one who is struggling feel success from the first two weeks. ‘So it makes everybody feel kind they’re on kind of an even playing ground’...