

# Reasoning with Functions

**F**UNCTIONS are one of the most important mathematical tools for helping students make sense of the world around them, as well as preparing them for further study in mathematics (Yerushalmy and Shternberg 2001). Functions appear in most branches of mathematics and provide a consistent way of making connections between and among topics. Students' continuing development of the concept of function must be rooted in reasoning, and likewise functions are an important tool for reasoning. Thus, developing procedural fluency in using functions is a significant goal of high school mathematics.

Key elements of reasoning and sense making with functions include the following:

- *Using multiple representations of functions.* Representing functions in various ways, including tabular, graphic, symbolic (explicit and recursive), visual, and verbal; making decisions about which representations are most helpful in problem-solving circumstances; and moving flexibly among those representations.
- *Modeling by using families of functions.* Working to develop a reasonable mathematical model for a particular contextual situation by applying knowledge of the characteristic behaviors of different families of functions.
- *Analyzing the effects of parameters.* Using a general representation of a function in a given family (e.g., the vertex form of a quadratic,  $f(x) = a(x - h)^2 + k$ ) to analyze the effects of varying coefficients or other parameters; converting between different forms of functions (e.g., the standard form of a quadratic and its factored form) according to the requirements of the problem-solving situation (e.g., finding the vertex of a quadratic or finding its zeros).

We address these key elements in more detail in the following sections.

## Using Multiple Representations of Functions

Different representations of a function—tables, graphs or diagrams, symbolic expressions, and verbal descriptions—exhibit different properties. Using a variety of representations can help make functions more understandable to a wider range of students than can be accomplished by working with symbolic representations alone (Lloyd and Wilson 1998; Coulombe and Berenson 2001). Students need to establish connections among different representations, for example, the relationship among the zeros of a function, the solution of an equation, and the  $x$ -intercepts of graphs.

Functions whose domains are the natural numbers are often represented recursively, where  $f(k + 1)$  is defined in terms of  $f(k)$  and an initial function value is given. Such functions can also be

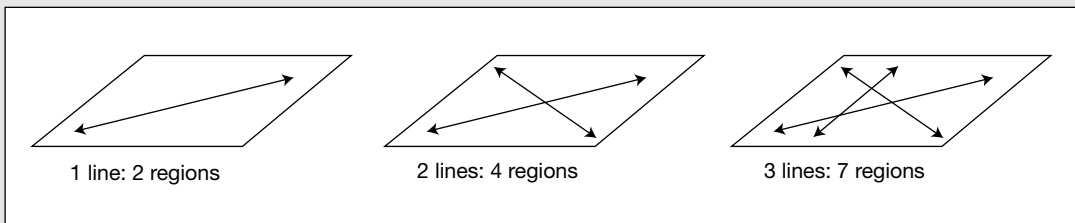
presented as sequences, and they are used in applications involving discrete rather than continuous data, often with the support of a calculator or an electronic spreadsheet; see example 10, “Patterns, Plane and Symbol,” and example 11, “Take As Directed.”

One interesting exploration is to investigate the number of regions into which a plane is divided by  $n$  straight lines. The first question is whether  $n$  lines always divide the plane into the same number of regions. Experiments with two or three lines should help students see that different numbers of regions result. For example, two lines that cross create four regions, whereas two parallel lines create only three. Three parallel lines produce four regions; three lines all going through the same point create six regions; but three nonparallel, nonconcurrent lines create seven regions. In general,  $n$  parallel lines create only  $n + 1$  regions, whereas  $n$  lines that all pass through the same point create  $2n$  regions. The next example explores what happens in the case of nonparallel, nonconcurrent lines. It could be used with students entering high school, who are beginning to develop the ability to express functional relationships symbolically, or with more advanced students who are working to improve their proficiency with algebraic manipulation. Example 10 illustrates how students can use various representations of functions, including verbal descriptions, tables, formulas, and various geometric models to make sense of a problem.

### Example 10: Patterns, Plane and Symbol

#### Task

Develop a symbolic representation for a function that produces the number of regions in a plane formed by intersecting lines such that no two lines are parallel and no more than two lines intersect in the same point, as shown in the figure.



#### In the Classroom

Students may approach this problem in several different ways, depending on their level of mathematical experience.

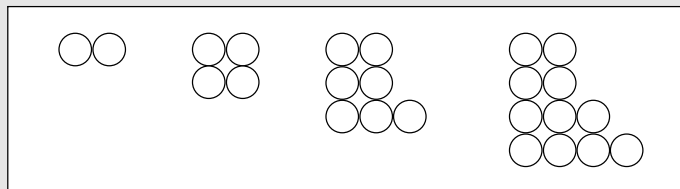
*Method 1.* After exploring a number of cases, possibly using an interactive geometry tool, students may produce a table of values for the number of lines,  $L$ , and the number of regions,  $R$ , as shown here:

No. of lines ( $L$ )	1	2	3	4	5	6
No. of regions ( $R$ )	2	4	7	11	16	22

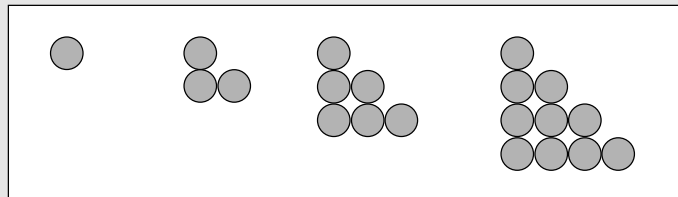
### Example 10: Patterns, Plane and Symbol—Continued

Students commonly observe the pattern of differences between consecutive terms of a sequence. This observation lends itself to a recursive definition of a function. In this example, a recursive definition would be  $R(1) = 2$ ,  $R(L) = R(L - 1) + L$ .

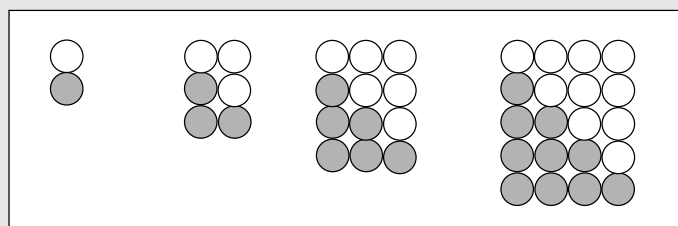
*Method 2.* An interesting geometric approach to helping students reason about this problem is to use checkers, coins, tiles, or other small objects to create a pattern in which the value of  $L$  defines the number of rows of objects and the value of  $R$  defines the total number of objects in the figure, as shown in the following sequence:



Removing one of the objects from the top row results in a triangular pattern, as shown below:



Doubling and transforming this pattern can result in a rectangle containing  $L(L + 1)$  objects, as shown here:



From this configuration, one can see that the original objects, minus the one that was removed, form half the rectangle. The explicit definition of this function can then be written as

$$R = \frac{1}{2}(L)(L+1) + 1 = \frac{1}{2}L^2 + \frac{1}{2}L + 1.$$

### Example 10: Patterns, Plane and Symbol—*Continued*

This type of approach demonstrates how a student who is given the opportunity to represent a function pictorially can use a simple geometric pattern to write the explicit definition of the function. The student can then extend his or her reasoning about symbolic representations by developing it from a familiar starting place.

*Method 3.* By applying technology to numeric and graphical reasoning, students may enter a number of ordered pairs from the table into a graphing calculator and examine a scatterplot of the pairs to conjecture that the relationship is quadratic because of the parabolic shape of the graph. On the basis of that conjecture, students could use calculator-assisted quadratic regression to find the function  $R = 0.5L^2 + 0.5L + 1$ . This algebraic model could be tested by using ordered pairs from the table.

*Method 4.* The teacher could ask students to focus on the differences between consecutive terms of the sequence of R values: 2, 3, 4, 5, and so forth. By applying algebraic reasoning, students may examine the data and observe that the function is quadratic because the first differences are linear so the second differences are constant. The geometric representation shown in part 1 of this example can be used to reinforce this concept for students. Writing a system of equations of the form  $R = aL^2 + bL + c$ , substituting three ordered pairs, and solving the system would reveal that

$$a = \frac{1}{2}, b = \frac{1}{2}, \text{ and } c = 1; \text{ thus}$$

$$R = \frac{1}{2}L^2 + \frac{1}{2}L + 1.$$

This strategy would require a sophisticated understanding of several mathematical topics and a command of algebraic manipulation developed later in the high school experience.

#### Key Elements of Mathematics

Reasoning with Functions—Using multiple representations of functions

Reasoning with Algebraic Symbols—Connecting algebra with geometry; Linking expressions and functions

#### Reasoning Habits

Analyzing a problem—seeking patterns and relationships

Seeking and using connections

Reflecting on a solution—justifying or validating a solution