Using multiple representations of functions

Functions are an indispensable object and tool of pure and applied mathematics. Mathematicians use a variety of kinds of functions: transformations, mappings, matrices, functors, recursive functions, parametric functions, piecewise functions, random variables, bijections, injections, isomorphisms, and homomorphisms. Mathematicians use various function representations to gain insight into mathematical systems and pure and applied problems.

There are a variety of function notations:

\[
\begin{align*}
 f(x) \\
y &= f(x) \\
f(x) &= \begin{cases} x^2, & x \leq 2 \\
 \sin(x), & x > 2 \end{cases} \\
f : x \rightarrow x^2 \\
x \xrightarrow{f} x^2 \\
vT & \text{ where } T \text{ is a linear transformation} \\
y &= x^2 \\
\begin{bmatrix} 2 & 5 \\
-1 & 3 \end{bmatrix}
\end{align*}
\]

A function can be represented symbolically, graphically, or as a table. Mathematicians move between these representations to determine different types of information. In a simple example, a symbolic representation like

\[
 f(x) = 2^x - x^2
\]
can be represented as a table or a graph.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5.00</td>
<td>-24.9688</td>
</tr>
<tr>
<td>-4.00</td>
<td>-15.9375</td>
</tr>
<tr>
<td>-3.00</td>
<td>-8.875</td>
</tr>
<tr>
<td>-2.00</td>
<td>-3.75</td>
</tr>
<tr>
<td>-1.00</td>
<td>-0.5</td>
</tr>
<tr>
<td>0.00</td>
<td>1</td>
</tr>
<tr>
<td>1.00</td>
<td>1</td>
</tr>
<tr>
<td>2.00</td>
<td>0</td>
</tr>
<tr>
<td>3.00</td>
<td>-1</td>
</tr>
<tr>
<td>4.00</td>
<td>0</td>
</tr>
<tr>
<td>5.00</td>
<td>7</td>
</tr>
</tbody>
</table>

From the symbolic representation, the function will be increasingly positive as \(x\) becomes large positively since for positive values of \(x\) the function given by the expression \(2^x\) is known to dominate any polynomial. The function will be increasingly negative as \(x\) becomes increasingly large negatively.
(x negative and $|x|$ large) since $2^x$ vanishes (approaches zero) for negative values whose absolute value is large. From the table, the function is increasing ever more rapidly as $x$ increases. This tabular information relies on the information already in hand from the analysis of the symbolic representation of the function. Finally, though the function graph looks like the graph of a cubic, we know from the analysis of the symbolic form of the function that it is not. The graph of the function on the given window clearly shows where the approximate values of the zeros and $y$-intercept of the function occurs, often important considerations in using functions.

In geometry, a transformation can be represented by an arrow as in the following diagram.

Here we can see that the translation transformation (function) takes a triangle onto a congruent triangle. (This diagram also gives a complete indication of what happens to every point in the plane.) Other mathematical relationships can be seen from the diagram as well in that parallel lines are mapped onto parallel lines and a line segment is mapped onto a line segment of equal length. The description of a translation as a rigid motion does not give the visual cues of a diagram showing a model of the translation. The symbolic representation of the translation $ABC \xrightarrow{\text{Translation}} A'B'C'$ does give a clear indication of what points are mapped onto what points, but it does not give the visual indication of the nature of this type of rigid motion.