PROMOTING EQUITY THROUGH REASONING

By Mary F. Mueller and Carolyn A. Maher

Learn how five characteristics of tasks and learning environments led these sixth graders to successful problem solving using direct and indirect reasoning to justify their solutions, make their justifications public, and respond to mathematical arguments.

Many educators share the vision of the Equity Principle—teachers holding high expectations for all students (NCTM 2000). However, according to National Assessment of Educational Progress (NAEP) reports, minority students continue to lag behind white students in mathematics achievement (Strutchens and Silver 2000). For example, on the 2000 NAEP mathematics assessment, 34 percent of white fourth graders scored at or above "proficient" compared to 5 percent of black students and 10 percent of Hispanic students (Braswell et al. 2001). Furthermore, the discrepancies are more pronounced on the extended, constructed-response items, which measure students' problem-solving and critical-thinking abilities (Kloosterman and Lester 2004). Several factors contribute to the failure of minority students to build meaningful mathematics learning in schools:

1. Low expectations for students' success in building conceptual mathematical knowledge
2. Classroom environments in which students are insufficiently challenged with thoughtful and engaging mathematical activities

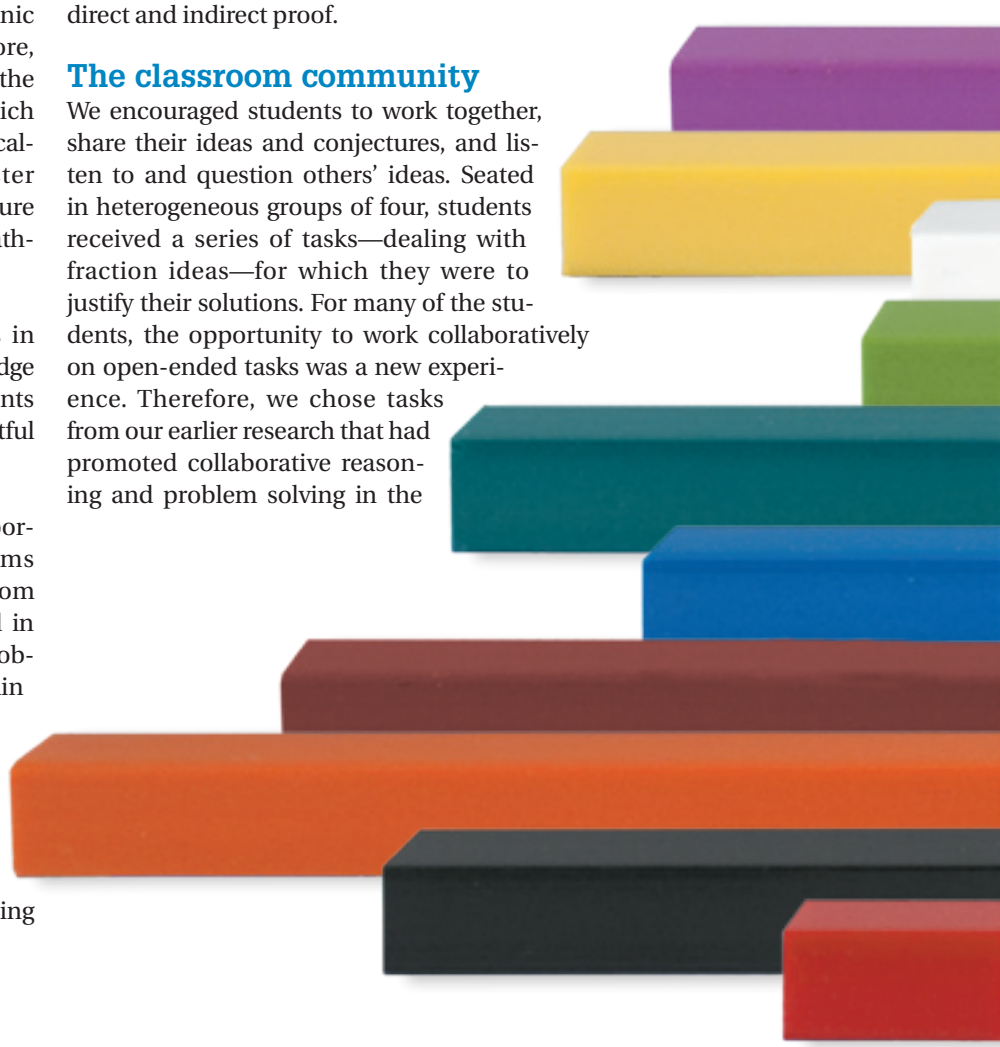
On the other hand, recognizing the importance of equitable practices in classrooms suggests optimism for achieving a classroom community where all students are engaged in meaningful and thoughtful mathematical problem solving. To accomplish this goal, certain classroom norms must be established in which teachers and classmates learn to listen to the ideas of all students and to recognize, respect, and value their contributions (NCTM 2008).

We implemented such equitable practices during an informal mathematics learning program. Twenty-four African American and Latino student participants volunteered to work on open-ended mathematical tasks as an extra after-school activity. Our strategies significantly engaged them in justification and reasoning during problem solving.

We offer two representative episodes to show how students' reasoning was made public in justifying problem solutions as well as in responding to and challenging the ideas of others. The sixth graders' reasoning took the form of both direct and indirect proof.

The classroom community

We encouraged students to work together, share their ideas and conjectures, and listen to and question others' ideas. Seated in heterogeneous groups of four, students received a series of tasks—dealing with fraction ideas—for which they were to justify their solutions. For many of the students, the opportunity to work collaboratively on open-ended tasks was a new experience. Therefore, we chose tasks from our earlier research that had promoted collaborative reasoning and problem solving in the



past. **Table 1** outlines the tasks that we used during five ninety-minute sessions. Students were given a set of Cuisenaire® rods (see **fig. 1**) and were invited to build models of their solutions. The set contains ten colored wooden or plastic rods that increase in length by increments of one centimeter.

After a problem was posed to the class, students had the choice to work alone or collaboratively. During their initial exploration, students worked in pairs and groups, while the teacher moved from group to group and observed students' activity, listened to their ideas and explanations, and encouraged them to continue their investigation. As appropriate, students were invited to share ideas with group members or prepare solutions for group sharing. Teachers' questions were designed to better understand students' thinking, to encourage students to talk about their ideas and work, and sometimes to direct students' attention to an incomplete component of an argument or extend their investigation about a mathematical idea. Teachers encouraged students to broaden their knowledge about an approach to a solution by listening to one another and considering ideas from others (Mueller and Maher 2009).

The sixth graders represented their solutions in various ways, often building models that they used to explain their ideas. These models helped communicate alternative ways of representing solutions to others. Students were encouraged to listen to one another when they judged their peers' arguments, interjected their opinions, and offered alternate solutions. Teachers did not judge or evaluate students' ideas, solutions, and strategies, so students did not fear being wrong. They were asked to justify their reasoning to their classmates and assess whether their solutions were convincing. Hence, students took responsibility for posing questions about ideas and evaluating the reasonableness of arguments.

Over the course of the five sessions, students engaged in high-order reasoning that often led to justifications in the form of proof. To convince their peers of an argument's soundness, they used multiple representations and forms of reasoning (see **table 2**). The following episodes provide examples of students' reasoning.

TABLE 1

Students received these challenges during the first five sessions of their after-school program.

Date	Tasks
11-12-03	1. If I give the yellow rod the number name *five*, what number name would I give the orange rod? 2. Suppose I give the orange rod the number name *four*; what number name would I give the yellow rod? 3. If I call the orange rod *one*, what number name would I give the yellow rod? 4. If I call the white rod *two*, what number name would I give all the other rods?
11-13-03	1. Suppose I call the dark green rod *one*; what number name would I give the light green rod? 2. Someone told me that the red rod is half as long as the yellow rod; what do you think? 3. If I call the blue rod *one*, find me a rod that would have the number name *one-half*.
11-19-03	1. Convince us that there is not a rod that is half the length of the blue rod. 2. Is *0.3* another name for the light green rod? 3. If I call the blue rod *one*, what number name would I give the white rod? What name would I give the red rod?
11-20-03	1. If I call the blue rod *one*, what number name would I give the red rod? What name would I give the light green rod? 2. If I call the blue rod *one*, what number names would I give the rest of the rods?
12-03-03	1. If I call the orange rod *one*, what number name would I give the white rod? What name would I give the red rod? 2. If I call the orange rod *ten* (fifty), what number name would I give the white rod? 3. I want to know which is bigger, one-half or one-third, and by how much.

FIGURE 1

A set of Cuisenaire rods contains ten wooden or plastic rods of specific colors that increase in length by increments of one centimeter.

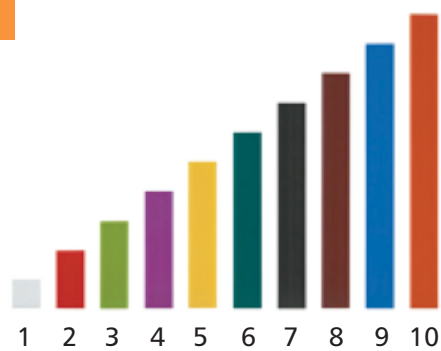


TABLE 2

Our study identified four forms of students' reasoning.

Form of Reasoning	Definition (for purposes of this study)
Direct	"A direct proof is based on the assumption that the hypothesis contains enough information to allow the construction of a series of logically connected steps leading to the conclusion" (Cupillari 2005, p. 12). Takes the form: $p \rightarrow q$
By Contradiction	Reasoning by contradiction, also known as the *indirect method*, is based on the agreement that whenever a statement is true, its contrapositive is also true or that a statement is equivalent to its contrapositive. For example, $p \rightarrow q$ is equivalent to (not q) $\rightarrow$ (not p); so if (not q) $\rightarrow$ (not p) is true, then $p \rightarrow q$ is also true (Cupillari 2005).
Using Upper and Lower Bounds	An *upper bound* of a subset S of some partially ordered set is an element that is greater than or equal to every element of S. The term *lower bound* of a subset S of some set refers to an element that is less than or equal to every element of S. An argument is then formed to justify a statement about the subset with the defined bounds (for example, that it is empty).
By Cases	For the purpose of this study, critical events were coded as *reasoning by cases* when students defended an argument by defending separate instances. Students defended an implication in the form $p \rightarrow q$, when p is a compound proposition composed of propositions $p_1, p_2, \ldots p_n$, and they established each of the implications $p_1 \rightarrow q$ $p_2 \rightarrow q$, $p_n \rightarrow q$.

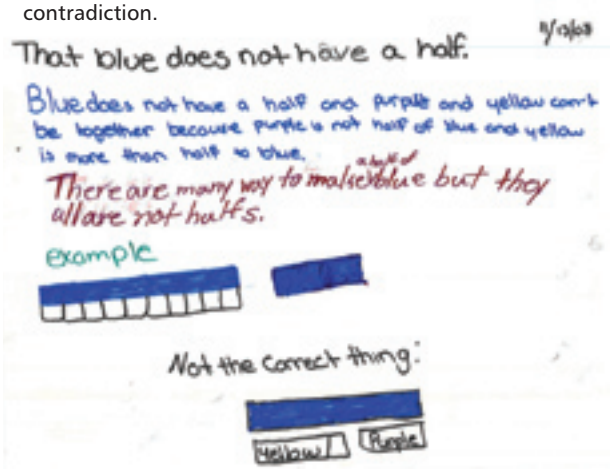
FIGURE 2

Small groups of students received a set of Cuisenaire rods and were invited to build models of their solutions.

(a) Shirelle and Michael erroneously called two rods of different sizes *one-half* the blue rod.

(b) Chris reasoned that the length of one blue rod equals nine white rods, which cannot be evenly divided in half.

(c) Chris defended his model with an argument by contradiction.



Episode 1

The problem was to find a rod with the number name *one-half* when the blue rod has the number name *one*. During the second session of the after-school math program, Shirelle and Michael proposed that the purple and yellow rods could serve as half the blue rod. They built a model showing the purple and yellow rods aligned with the blue rod and used direct reasoning to support their conjecture (see **fig. 2a**).

Dante challenged their claim, arguing that the purple and yellow rods are not the same length, so they could not be called *halves*.

Using a counter argument, Chanel explained, "And the yellow takes up more space than the purple; to be halves, they should be the same."

At another table, Chris proposed to his group that no rod in the set could be called *one-half*. He reasoned that nine white rods are equivalent to the length of the blue rod (see **fig. 2b**) and cannot be partitioned into two sets without a remainder. Chris defended his model using an argument by contradiction (see **fig. 2c**):

> Um, like y'all was saying, the white little rods won't be able to do it, but since there's nine white little rods, you can't really divide that into a half. So, you can't really divide by two because you get a decimal or remainder. So, there is really no half, no half of blue because of the white rods.

Also using a model (see **fig. 3**), Dante showed that the purple rod could not be considered half of the blue rod because the combination of two purple rods is not equivalent to the length of the blue rod (they are too short). Likewise, the yellow rod could not be named *half* of the blue rod because the combination of two yellow rods was not equivalent in length to the blue rod. Dante's argument uses upper and lower bounds.

Justina presented an alternative argument based on portioning the set of rods into two cases, those with halves and those without (*singles*), enumerating all the cases (see **fig. 4**).

Students used four different forms of reasoning to justify their solutions for this problem (see **table 2**). In only one case was the reasoning faulty (Michael and Shirelle's direct argument that the yellow and purple rods could both be called *half* the blue rod). As early as the second session, students listened to one another's ideas and proposed arguments for the reasonableness of their own solutions.

FIGURE 3

Dante offered an argument using upper and lower bounds.

FIGURE 4

Justina presented an argument based on portioning the set of rods into two cases: those with halves and those without.

Episode 2

During session 3, one student, Jeffrey, posed the task of naming the red rod when the blue rod is named *one*. At the beginning of session 4, students had an opportunity to build on their earlier problem solving. Individual groups of students initiated the challenge of naming all the rods (when the blue rod is named *one*).

At one table, Chanel named the remainder of the rods using direct reasoning based on the incremental increase by one white rod, or one-ninth. She used the staircase model (see **fig. 1**) as a guide and named the rods, increasing by one-ninth, hesitating at the orange rod, and then naming it *rod nine-tenths*. When she explained her dilemma—of naming the orange rod—to Dante, he initially named the orange rod *ten-ninths* but corrected himself and said that the orange rod would start a "new cycle" and be named *one-tenth*. Dante told the group that he heard students at other tables calling the orange rod *ten-ninths*. Michael insisted that the others were incorrect. Chanel agreed and claimed that "the denominator can't be smaller than the numerator." Dante concurred. They discussed a rule, which Michael referred to as *the laws of math* and *the laws of facts*, that states that the denominator cannot be smaller than the numerator.

Reminded by a teacher that the white rod is named *one-ninth*, Dante finally used the staircase model to name the orange rod *ten-ninths*. He explained that the length of ten white rods is equivalent to the length of an orange rod and that because a white rod is called *one-ninth*, the orange rod will be called *ten-ninths*. Later

during the same session, students shared their findings with the whole class (see **table 3**). Five students presented arguments using direct reasoning. However, they based their arguments on different representations.

A culture of confidence

In a relatively short period of time, a culture of sense making, communication, and collaboration evolved over the five sessions. The first episode occurred during session 2 of the program. Students were already working together to build representations, questioning each other, and defending their solutions. Chanel and Dante collaborated in an attempt to convince Michael and Shirelle that the purple and yellow rods could not be called *one-half*. They built an argument using upper and lower bounds. During whole-class sharing, both students presented sophisticated versions of their argument.

The second episode began with students attending to the misconception that the numerator is always less than the denominator. By building a model, they convinced themselves and one another that the blue rod could indeed be named *ten-ninths*. Students were confident in sharing their justifications and secure with having representations that differed from their peers' models.

A culture of equity

Educators often suggest that minority-race, inner-city students "need structure." Adults therefore organize classrooms to focus on having students learn procedures and skills. This perspective leaves little room for students to reason, conjecture, and share ideas. As a result, students develop a view of mathematics as rule oriented and procedure driven (Powell 2004).

In contrast, our after-school informal math sessions focused on having students build personal meaning of mathematical ideas. As the above episodes illustrate, students actively engaged in solving problems and justifying their solutions. They posed arguments and defended those arguments. They questioned and corrected one another and ultimately created justifications that took the form of proofs. We documented four types of reasoning that students used to defend their arguments. Students used multiple representations to back up their claims and convince their classmates.

Putting dispositions into practice

Teachers can promote such sense making and reasoning by engaging students in similar activities in their own math classs. Give your students responsibility for justifying their problem-solving solutions. To encourage teachers, we share certain characteristics of the tasks and environment that led to successful problem solving:

1. Give choices. Seat students in small groups; participants then have options: to work

TABLE 3

Students were asked to share their direct arguments for naming the orange rod *ten-ninths* with the whole class.

Student	Direct Argument
Lorrin	Before, we thought that because we knew that the numerator would be larger than the denominator, and we thought that the denominator always had to be larger. But we found out that was not true because two yellow rods equal five-ninths and five-ninths plus five-ninths equal ten-ninths.
Kia-Lynn	The orange [rod] is bigger than the blue one, but when you add a one-ninth—a white rod—to the blue top, it kind of matches. We found out that you can also call the blue rod *one and one-ninth* and the orange *one* without the one-ninth; without the white rod is also called *one-ninth*, too. If you have one white rod and you add it to the blue, it's *one-ninth plus one* [or] *one and one-ninth*. So, if the blue rod and one white, if you put them together, then this means that it's *ten-ninths* also known as *one and one-ninth*.
Dante	Well, all I did was start from the beginning—start from the white—all the way to the orange and—like Kia-Lynn's group just said—I had found a different way to do it. Because all I had used [was] an orange, two purples, and a red. Since these two are purple, and this is supposed to be purple, but I had purple, and I used a red since four and four are eight, so, which will make it *eight-ninths* right here. And then plus two to make it *ten-ninths*. That's what I made.
Chanel	For all of these, I gave the white rod *one-ninth*, the red rod *two-ninths*, the light green rod *three-ninths*, *four-ninths* for the pink—uh, purple—rod, *five-ninths* for the yellow rod, *six-ninths* for the dark green rod, *seven-ninths* for the black rod, *eight-ninths* for the brown rod, *nine-ninths* for the blue rod, and for orange I gave it *one-ninth*.
Chris	I was saying *one and one-ninth* because if you add one blue one and since the number name for the blue is *one*, then if you add a white one, that equals *one-ninth*. If you add *one* and *one-ninth*, then that would equal *one and one-ninth*.

individually, with a partner, with a subset of the group, or with the entire group. Thus, students may engage in the way in which they learn best.

2. Differentiate. Appropriate teacher moves can facilitate how students share ideas. For example, to promote interest in the ideas of others, ask a student if he or she is aware of another student's solution. Point out different and contradictory claims and leave it to the students to work out the reasonableness of the arguments; this encourages them to share ideas and listen to one another. Allowing adequate time to explore, share, and revisit problems will respect the pace of slower-working students who may have different learning styles. Having extension tasks available is essential to challenging those who work more quickly.
3. Make ideas public. After students explore in their small groups, invite them to use an overhead projector to write ideas and share solutions. Have them make a variety of representations public and discuss them. Organize the order of the presentations by asking several students to share their solutions and strategies so that it becomes apparent that alternate paths and ways of representing mathematical ideas exist rather than only one "correct" way. Emphasize the importance of offering arguments that are convincing to classmates, not just to the teacher. When conflicts arise, ensure that ideas are public; a resolution can be postponed until convincing evidence is offered. Communicating their ideas will engage learners. Taking responsibility for explaining ideas can lead to increased student confidence and autonomy.
4. Select the best tasks and tools. Choose open-ended tasks that allow for multiple entry points at multiple levels. All students can work from their personal representation and form their own ideas from this starting point; thus, all students can realize success from the onset. Opt for tasks that are novel to the students so they do not have a strategy readily available. Then they must rely on their own (and their partners') resources to plan a strategy and build new knowledge.

   Revisit tasks or pose similar tasks so that students have time to reflect on their previous solutions and those of their classmates and incorporate these strategies into their solution, thus promoting more refined justifications.

   Have manipulative materials available and encourage students to build models to show their conjectures. Urge them to record their strategies with pictures, numbers, and words.
5. Hold high expectations. Students' mathematical development is crucially linked to our aspirations for them. We need strategies for putting our high expectations into practice. Helping students to develop a culture of reasoning supports them in meeting rigorous standards and working up to their mathematical potential.

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