Wheatley, Grayson H. *Effective Learning Environments for Promising Elementary and Middle School Students*. Developing Mathematically Promising Students (April 1999): 71-80.

Chapter 6

Part II

EFFECTIVE LEARNING ENVIRONMENTS FOR PROMISING ELEMENTARY AND MIDDLE SCHOOL STUDENTS

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As you enter Ms. Diaz's classroom, you see Sara and Tom engaged in a heated exchange about the interpretation of a problem they are solving while the other students are solving the same or other problems, also in pairs. Sara says there can be many possible answers, whereas Tom maintains that there is only one. Tyrone and Marta are jointly collaborating on developing a solution to the same problem, playing off each other's ideas. Ms. Diaz is moving around the room being very attentive to the interaction but not becoming too engaged with any group. After twenty-five minutes, she calls the class together and asks Sara and Tom to explain their reasoning for the problem they solved. The other students had worked on the same task, perhaps at a different time. As Sara and Tom explain their solution, some students express disagreement and others ask for clarification. Throughout their explanation, it was clear that Marcie was eager to tell about her method. It is a lively discussion, with Ms. Diaz listening and intervening to facilitate the interactions.

As described in the *Professional Teaching Standards* (National Council of Teachers of Mathematics 1991), if students are to develop mathematical power, major changes in instruction are needed. Central to the shift envisioned by the *Professional Teaching Standards* authors is a vision of social norms through which classrooms become mathematical communities with all that is implied by that characterization. In mathematical communities, individuals assume responsibility for their actions and statements, with reasoning and mathematical evidence as a test for viability rather than the teacher acting as the mathematical authority (Cobb, Wood, and Yackel 1995). This call for change is broad based. Along with selecting appropriate tasks for students constructing knowledge, negotiating a classroom culture conducive to learning is highly important (Steffe 1990).

NEGOTIATING SOCIAL NORMS

Students develop a set of expectations about each class they experience. These expectations are based on a set of social norms. The way each student acts in a class influences the microculture. A skillful teacher will negotiate rather than impose ways of interacting in the class. In some classrooms, students expect to sit quietly, not talking except to answer questions posed by the teacher and then only when recognized. In such classes, the structure imposed by the teacher dictates how students are to act, and failure to comply with the rules results in disciplinary action. The norms in other classes might include the belief that joking around and distracting class activities is a goal. Such norms are obviously counterproductive.

Social norms are defined as those ways of acting and interacting that are accepted by the group and that guide action. One example of a more positive norm would be students accepting the obligation of making sense of classmates' mathematical explanations and questioning those explanations when necessary. Other social norms that might become part of the praxis of a productive class are: (1) Students expect to be challenged and develop their own solutions rather than following a set procedure demonstrated by the teacher; (2) students expect to construct their own methods and recognize that different students will carry out an operation in different ways; (3) students expect to collaborate with other students and serve as a community of validators; and (4) students expect the process of solving a problem to take time and reflection.

In addition to the social norms just described, it is important to negotiate sociomathematics norms (Yackel and Cobb 1996). By *sociomathematics norms,* Yackel and Cobb refer to "what counts as a mathematics justification" and "why some student explanations are 'better' (we might say more elegant) than others." Both of these sociomathematics questions are part of what it means to do mathematics. It is promising students who are most likely to appreciate elegant solutions and strive toward innovative ways of thinking.

CLASSROOM CULTURES

Although the teacher plays a role in creating the learning environment, he or she alone does not establish it (Bauersfeld 1996; Varela, Thompson, and Rosch 1993). Rather, a classroom culture is coconstructed by all the participants. The learners both contribute to and are strongly influenced by the culture of the classroom. The nature of each individual's participation in this construction is a subtle and complex activity. The presence of just one person shapes the culture. The teacher may play an important role in negotiating an envisioned learning environment, but the classroom culture that results is strongly influenced by the students themselves. The reality of the classroom is continually developing and being interactively constituted (Bauersfeld 1980). Social norms for promising students should include the valuing of creativity and the joy of doing mathematics.

In some classes, students enthusiastically participate in making meaning by questioning peers, developing their own methods, and justifying their explanations in the process of constructing their mathematics. In other classes, students see their role as following directions, carrying out procedures in prescribed ways and relying on the teacher as the source of knowledge. When instruction is skill based and rule governed, many students, in self-defense, adopt a "play the game called school" stance, attempting to decide what is required to get a good grade rather than forming the intention of making sense. The classroom culture is a major determiner of which stance students take. When the culture of the classroom encourages students to inquire, question, conjecture, collaborate, and evaluate, students will learn more mathematics than if they are required to listen to teacher explanations, complete practice exercises in a prescribed way, and rely on the teacher to know if their answers are correct.

Bauersfeld (1996) contends that learning is *in* the interactions of individuals and therefore gives importance to the culture of the classroom. For him, a culture where students are encouraged to construct meaning for themselves through interactions with others is essential. If there are no student-to-student interactions, then the learning environment is impoverished. Thus, establishing a learning environment in which challenging others' mathematical reasoning is viewed as constructive and positive translates into rich learning opportunities for promising students. Student-to-student interactions can play an important role in how promising students become mathematically powerful.

PROBLEM-CENTERED LEARNING

One instructional model that has proved effective with promising students is problem-centered learning (Wheatley 1991). In problem-centered learning, the class begins with a problem posed by the teacher—or perhaps by a student. For example, the teacher might pose the following problem to a second-grade class: "I have a pocket full of just pennies, nickels, and dimes. If I reach in and pull out three coins, how much might the three coins be worth? Try to find all the possibilities." The class is then organized into small groups (two or three students of similar capabilities), and the students work collaboratively in their groups on the tasks posed.

After about twenty-five minutes, the students are assembled for class discussion. Students present their solutions to the class for consideration by the group, which then serves as a community of validators. During the class discussion the teacher is nonjudgmental and the viability of solution methods is determined by the class, not the teacher. In problem-centered learning the teacher has four main roles: selecting appropriate tasks on the basis of his or her knowledge of the students, organizing the groups, listening carefully as groups work, and finally, facilitating the class discussions. The success of this model is dependent on the classroom culture and the social norms that have been negotiated.

Tasks

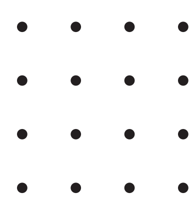
Promising students thrive on intellectual challenges. Many promising students who languish in a skills-based mathematics program come alive when given the opportunity to pursue nonroutine mathematics problems. Some of our most promising students turn away from mathematics because they see it as a set of procedures to be done in a prescribed way. Investigations for promising elementary and middle school students should have the following characteristics:

- Be potentially meaningful to students;
- Be problem-based;
- Be replete with patterns;
- Encourage students to make decisions;
- Lead to rich mathematical experiences; and
- Promote discussion and communication.

Example

Inspired by Kennedy's article in the *Mathematics Teacher,* the following week-long activity was designed (Kennedy 1993). Annette Smith, a middle school teacher, used this activity with a seventh-grade class. Students were provided with one-inch dot paper and organized in groups to work on the following problem.

> Draw all possible noncongruent triangles that have their vertices on the dots on the four-by-four square grid. How many are there? How could you convince someone you have them all? (You may want to stop reading and try to draw all twenty-nine possibilities.)



Students were asked to draw triangles on the dot paper, cut them out and post them on the bulletin board. Duplicates, if students spotted them, were removed when agreement was reached that they were indeed congruent. Examples of the triangles posted are shown in figure 6.1.

To draw all possibilities, students had to develop a system for classifying the triangles. This generated much discussion with a variety of classification plans used. The triangles on the bulletin board were often rearranged by the students to help develop patterns. Each day, Mrs. Smith would have a class discussion in which students would describe their reasoning and conjectures. This also generated much discourse. The classification of triangles by sides and angles was a topic of interest for several days. Students were surprised they could not draw an equilateral triangle on the four-by-four grid.

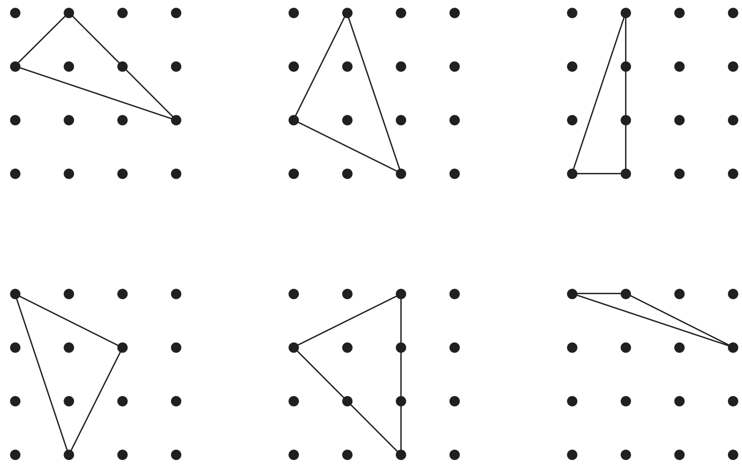


Fig. 6.1. Examples of triangles formed on a 16-point grid

Mrs. Smith had students keep portfolios of their investigations and asked them to write descriptions of their activities and what they learned. Students were encouraged to pose and investigate additional questions in this setting. Some students wondered how many triangles there would be if the grid were five-by-five. Others attempted to determine all quadrilaterals that could be formed on a four-by-four grid. (There are many!)

Extensions

Other questions that were raised included:

- What perimeters are possible? What is the largest perimeter? Smallest? (This led to a discussion of ways to find the distance between two points, the Pythagorean theorem, and square roots.)
- What areas are possible? What is the largest area possible? The smallest?
- What angle measures are possible? What is the largest angle measure? The smallest?

This activity involved seminal mathematical concepts such as systematizing, transforming, and classifying. The task seemed simple—just draw triangles—but it proved to be quite challenging. Students had to develop a way of systematically classifying the triangles to know when they had them all. They had to make many decisions. The activity promoted much discussion, and students were challenged to investigate related, deeper mathematical concepts. The investigation was replete with patterns and involved fundamental mathematical ideas.

In addition to investigating geometric concepts, it is important for promising intermediate and middle school students to learn to reason proportionally. Two proportion tasks that have proven effective follow.

1. A photograph that is 6 inches high and 4 inches across the bottom is to be enlarged so that it will be 6 inches across the bottom. How high will it be? (Many students give 8 as an answer, adding 2 because 6 is 2 more than 4.)
2. A videotape can record 2 hours on short play and 4 hours on long play. After 37 minutes have been recorded on short play, the recorder is switched to long play. How many minutes can then be recorded on long play?

Collaboration

Differing patterns of interaction evolve among students in small groups. If one of the students in a pair becomes a mathematical authority—that is to say, the other student does not question the validity of what he or she does and defers to that person for decision making—little or no learning occurs (Cobb 1995). This finding conflicts with the folk wisdom of pairing a "good" student with a weaker one. The evidence is strong that pairs should be formed of students who will challenge each other. It is the resolving of the perturbations resulting from disagreements that produces learning. Thus, we should attempt to pair individuals who will challenge each other's thinking as they attempt to give meaning to their mathematical experiences.

Class Discussion

If students interpret the environment as a recitation, then the way they act will reflect that interpretation and they will see themselves in an evaluative position. In contrast, we can negotiate a classroom environment that is interpreted by students as a sense-making place where their ideas are valued and listened to. When classrooms are seen as learning places, rather than work places, the dynamics can foster risk taking and result in lasting learning.

The belief must be fostered that a peer has something to say that is worth listening to. But this fostering is a delicate matter. For example, if student utterances are filtered through the teacher, students do not develop a sense of communicating with one another. Repeating student comments "so all can hear" will discourage true discourse and the building of community.

Students learn in many different ways, and we must be supportive of the individual differences. Some very capable students are "active" listeners in class discussions. They do not listen easily when sitting still and looking at the speaker but understand what is being said better when they are in motion. Some individuals think best while they are physically moving or doodling. Krutetskii (1976) reports that when he gave a task to a five-year-old girl, she got up, did a somersault, sat back down and wrote the answer. Although students moving around and perhaps making utterances can be distracting, we may want to explore ways of allowing certain students to be physically active during class discussion.

As another example, during class discussion Brad, a third grader, was observed playing with pencils and not looking at the speaker (Lo, Wheatley, and Smith 1994). Afterward, in a video-recorded interview, he could describe what the speaker said, whether it made sense, what her intentions were, what other students were doing, and how the teacher was reacting to the class. Brad could actually listen better when he was active. Some adults have the practice of doodling while listening to a speaker and claim they can attend more easily that way. Thus "paying attention" is not a prerequisite for meaningful participation in class discussion (Langer 1997).

CHARACTERISTICS OF PRODUCTIVE CLASSROOMS FOR PROMISING STUDENTS

According to Maker (1996, p. 31), learning environments for promising students should have the following characteristics:

- Learner-centered rather than teacher- or content-centered
- Independence rather than dependence emphasized
- Open rather than closed to new ideas, innovations, explorations
- Acceptance rather than judgment exercised
- Complexity rather than simplicity as a focus
- Varied groupings rather than one grouping as a general organization
- Flexibility rather than rigid structure or chaotic lack of structure
- High mobility rather than low mobility permitted and encouraged

To Maker's list, I would add that it is important for promising students to be in a setting where the instruction is fast paced. Such an environment encourages students to become mentally agile, to respond quickly, and thus to develop a brain that responds rapidly. One intermediate teacher, Mrs. Joyner, used time that might be lost in transition from one activity to another to present mental mathematics questions such as "What is 21 + 19?" or "What is $100 \times 7 - 250 + 5000$?" or "How many cubes of whole numbers are there less than 100?" Note that the choice of tasks encouraged students to construct number patterns and build relationships among numbers.

The study of mathematics for promising students should be fast paced and problem centered, focusing on concepts rather than procedures. Algebra texts usually have sequential lessons where students plot points in a plane, make tables, and then sketch linear equations. Promising students respond well to lessons using graphing calculators where quadratic, cubic, and linear equations are dealt with conceptually—all in the same lesson—through the posing of problems. By considering these different types of equations together, students gain a broader perspective on functions. This has quite a different effect from a typical beginning algebra course that would have several lessons just on procedures for solving and graphing linear equations. Lessons that consider concepts and relationships encourage the construction and interrelating of powerful mathematics and appeal to promising students.

Too often mathematics instruction has emphasized computational procedures, such as borrowing in subtraction of whole numbers and long division, rather than patterns and relationships. I recently observed an eighth grade class of promising students solving equations. A procedure to be followed was demonstrated, and twenty-four problems, all of the form $ax = b$, were assigned. For example

1. $-4x = 20$ 2. $6x = -48$ 3. $-8x = -56$ 4. $-3x = 39$

These promising students were instructed to show on paper the multiplication of both sides of the equation by the multiplicative inverse of a and, of course, check their solutions. The effects were stultifying. Students were confused, frustrated, and certainly not pleased with their experience. They saw no reason to adhere to a complex written procedure when they could determine the solution by inspection. In contrast, another teacher, Mr. Santos, formed the class of thirty students into pairs and asked them to determine the values of x that satisfied each equation shown below. No procedures were demonstrated and students were not expected to use any particular method.

1. $2x - 5 = 7$ 2. $3x + 4 = 20$ 3. $21 = 7 - x$ 4. $x^2 = 6$

After students had discussed these problems in their pairs and devised their own methods, they came to the front of the room and explained their thinking to the class. Varieties of solutions were explained for each of the four different types. For example, one student explained her solution method for number four by saying, "I saw that 30 divided by 5 was 6, so $x - 3 = 30$. Then it was easy to see that $x = 33$ since $33 - 3$ is 30." Students were eager to explain the creative ways they had devised for solving the problems and showed much interest in the methods used by others. Everyone was required to justify his or her reasoning to the class. There was an intention of making sense. These problems could be followed by others that would raise their mathematical reasoning to an even higher level such as $3x^2 - 6 = 42$, $x^2 + x + 3 = 23$, or $x^2 + 1 = 0$.

"Fast-paced" should not be interpreted as short-answer, skill-based lessons. Mathematics tasks should require more than a few minutes to solve. For example, middle school students could be challenged to find how many numbers from 1 to 180 inclusive have 2, 3, or 5 as factors. In solving this problem, students might need considerable time to explore and search for patterns, consider alternative methods, try several approaches, make generalizations, and determine the validity of their answer. Thus, fast-paced lessons should feature challenging tasks.

ENCOURAGING CREATIVITY

It is vital that the teacher negotiate a set of social norms that foster curiosity, creativity, and sense making. For example, beginning a class with a challenging problem is one way of getting students involved in thinking mathematically. However, before discussing the value of such an approach, it is important to think about what is meant by a *problem.* To design an appropriate task, the teacher must have some sense of what each student knows about the topic so that a task can be designed that is within her or his zone of potential constructions— it is challenging but possible. Once students have become engaged in a task, curiosity and creativity can be encouraged by extending the problem, by asking "what if" questions and encouraging students to formulate related problems. As Brown and Walter (1983) point out, having students pose problems takes mathematics to a higher plane, a plane where they are acting mathematically in quite powerful ways.

SUMMARY

The dynamic culture of a mathematics classroom for promising elementary and middle school students is an important influence on the nature of mathematics learning. As a teacher develops a vision of mathematics classrooms compatible with the recommendations of NCTM (1991), steps can be taken to negotiate a culture of the classroom that encourages intellectual autonomy, curiosity, and sense making. A classroom culture will develop, and the teacher can play a vital role in negotiating, not legislating, a learning environment that encourages promising students to become mathematically powerful (Cobb et al. 1988). The social norms that come to be established in a class will constrain or facilitate the mathematics learning of promising students.

One model of instruction that has proven effective is problem-centered learning (Nicholls et al. 1991; Wood and Sellers 1996). Problem-centered learning is designed to encourage promising students to construct knowledge for themselves in ways that the learning will be lasting. As students work collaboratively on challenging tasks and have their ideas tested in an intellectual community, they develop confidence and knowledge. The explain-practice method of teaching has serious weaknesses and is inappropriate for promising students. New technologies allow attention in mathematics classes to be focused on concepts rather than procedures, thus freeing students to consider more challenging and meaningful tasks.

As we learn more about classroom cultures and how they form, we are in a better position to design more effective learning environments that empower promising students. For too long, schools have encouraged a transmission view of teaching. More recent theoretical and practical studies have shown that knowledge cannot be transmitted but that meaning is evoked by each individual's experiences in a learning setting. By encouraging promising elementary and middle school students to construct mathematical patterns and relationships, we can help them become powerful learners for a lifetime.

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