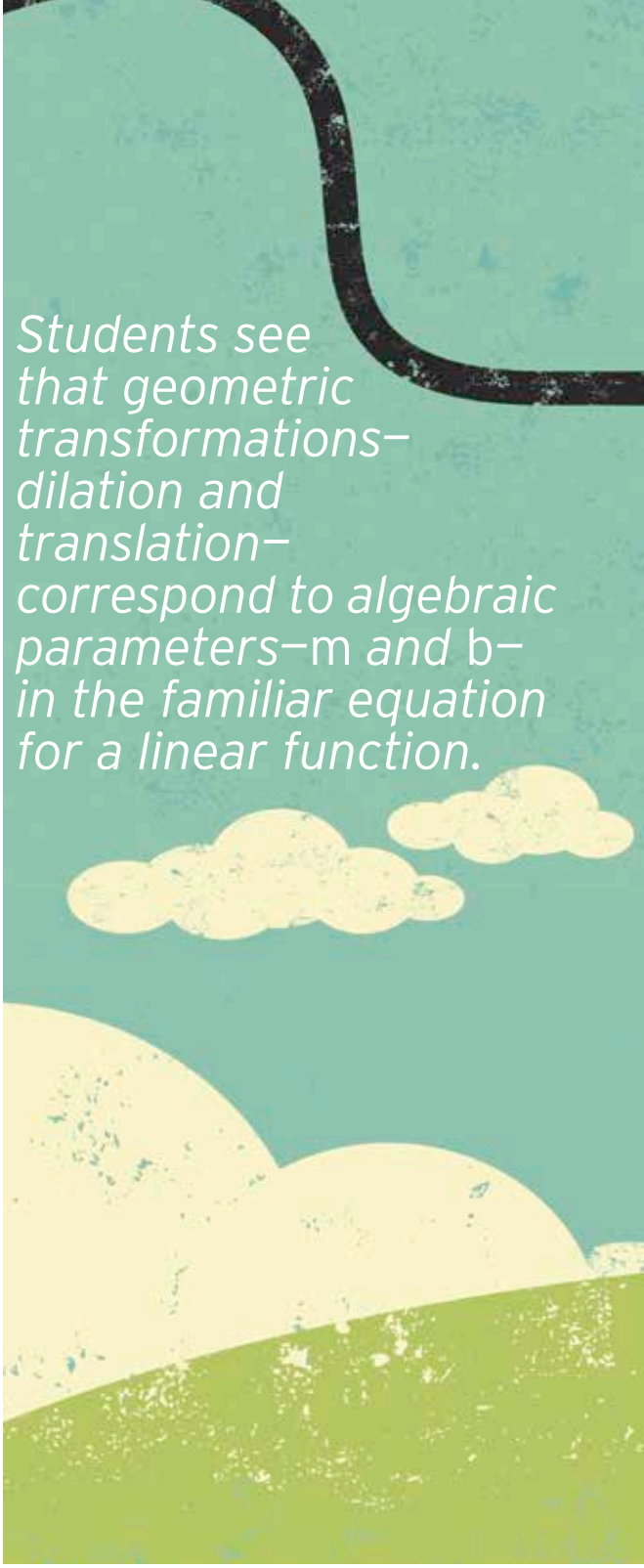
CONNECTING FUNCTIONS in Geometry and Algebra

Students see that geometric transformations—dilation and translation—correspond to algebraic parameters—m and b—in the familiar equation for a linear function.



RETROROCKET/GETTY IMAGES

Scott Steketee and Daniel Scher

One goal of a mathematics education is that students make significant connections among different branches of mathematics. Connections—such as those between arithmetic and algebra, between two-dimensional and three-dimensional geometry, between compass-and-straightedge constructions and transformations, and between calculus and analytic geometry—form the backbone of important mathematical understandings.

Another such connection is described in the Common Core State Standards for Mathematics: Students should "describe transformations as functions that take points in the plane as inputs and give other points as outputs" (Standard G-CO2, CCSSI 2010, p. 76). The short length of this quotation belies its importance in relating geometric transformations and functions. Both topics are emphasized in the Common Core: Transformations are the conceptual basis for congruence and similarity, and functions get an entire conceptual category of their own.

If our response to having students "describe transformations as functions" is limited to explaining how transformations fit the abstract definition of function, then students miss an opportunity for fruitful sensory-motor interactions with geometric functions. (We use *geometric functions* to refer to geometric transformations treated as mathematical functions.) If students have no meaningful way to connect, for instance,

dilations and translations in the geometric realm with linear functions in the algebraic realm, the connection between geometric and algebraic functions will be a factoid, a bit of trivia without real value.

In this article, we describe a way of forging a strong connection between geometric and algebraic functions, a connection that can deepen students' concept of function and develop their appreciation for the interconnectedness of geometry and algebra.

The activities described in this article (and additional geometric functions activities) are freely available in classroom-ready form, including student worksheets and teacher notes. Each activity is supported by an interactive Web page (http://geometricfunctions.org/links/connecting-functions), powered by Web Sketchpad®, that runs in any modern Web browser. No other software is required.

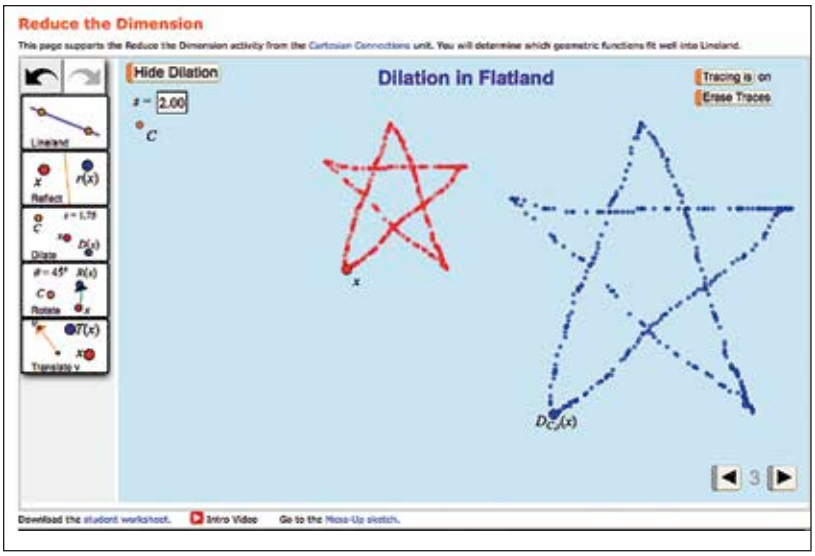


Fig. 1 A worksheet at http://geometricfunctions.org/links/connecting-functions guides students to create a Web sketch.

Each activity page includes a Web sketch (as illustrated in **fig. 1**), customized construction tools, a scrolling copy of the worksheet, links to a downloadable worksheet, brief video clips, and other support materials.

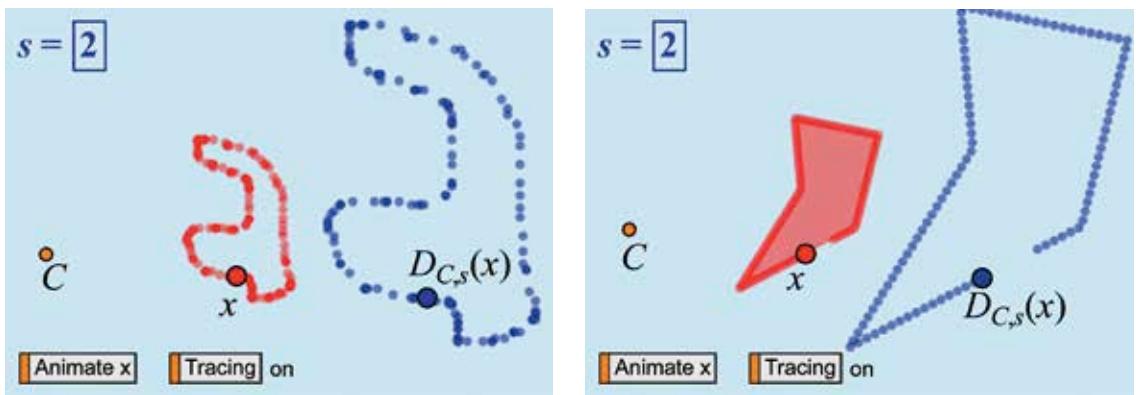
A GEOMETRIC FUNCTIONS APPROACH

The textbook *Geometry: A Transformation Approach* (Coxford and Usiskin 1971) broke new ground in cultivating the connections between transformations and functions. Now, with the availability of dynamic mathematics software such as The Geometer's Sketchpad®, Cabri Geometry®, and GeoGebra, the benefits for students' understanding of function concepts are stronger still.

Cognitive scientists tell us that students build abstract mathematical concepts by connecting those concepts to the physical world through *conceptual metaphors* (Lakoff and Núñez 2000; Radford 2012), such as the metaphor that numbers are points on a line. Geometric functions are based on a similar metaphor—that geometric variables are movable points. (For the purposes of this article, we will consider movable points on the plane.) This metaphor enables students to use dynamic software to create a point (the independent variable), construct another point (the dependent variable) that depends on the first, and drag to observe the resulting covariation and relative rate of change. In other words, a geometric function relates the preimage point—the independent variable x—with its image—the dependent variable that is a function of x.

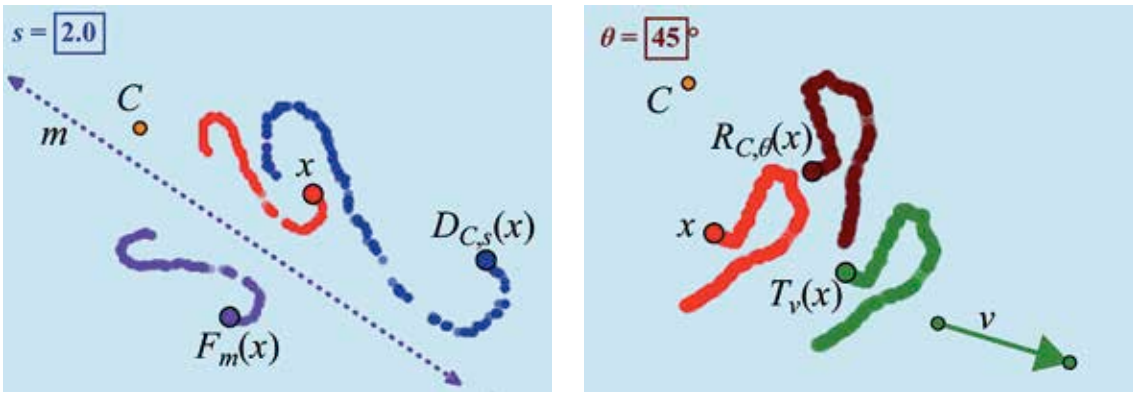
Computer-based kinesthetic and visual encounters with geometric functions make this and other related metaphors real to students and enable them to build abstract function concepts based on their sensory-motor experiences. For instance, in **figure 2a**, the student uses Web Sketchpad to create a point (independent variable x), to construct its image under a dilation, and to label the image using meaningful function notation. She then drags x while observing that the dependent variable moves in the same direction as the independent variable but at twice the speed. In **figure 2b**, she restricts the domain to a polygon and again drags to observe the corresponding range. (In the Web version of this article, this and all following figures are interactive; readers can drag the variables, edit the parameters, and press the buttons.)

Continuous dragging in a dynamic geometry environment is an innate feature of the software but one that is often missing from students' manipulation of symbolically expressed functions. Dragging variables can be particularly important in building students' understanding of Cartesian graphs, which students often comprehend as static pictures (Cuoco and Goldenberg 1997; Hazzan and Goldenberg 1996).



(a) (b)

Fig. 2 The label $D_{C,s}(x)$ stands for the dilation of x about center point C by scale factor s.



(a) (b)

Fig. 3 Students use meaningful function notation whenever possible. $F_m(x)$ is the flip (or reflection) of x across mirror m (**a**); $R_{C,\theta}(x)$ is the rotation of x around C by angle θ; and $T_v(x)$ is the translation of x by vector v (**b**).

CONTROLLING THE INPUT

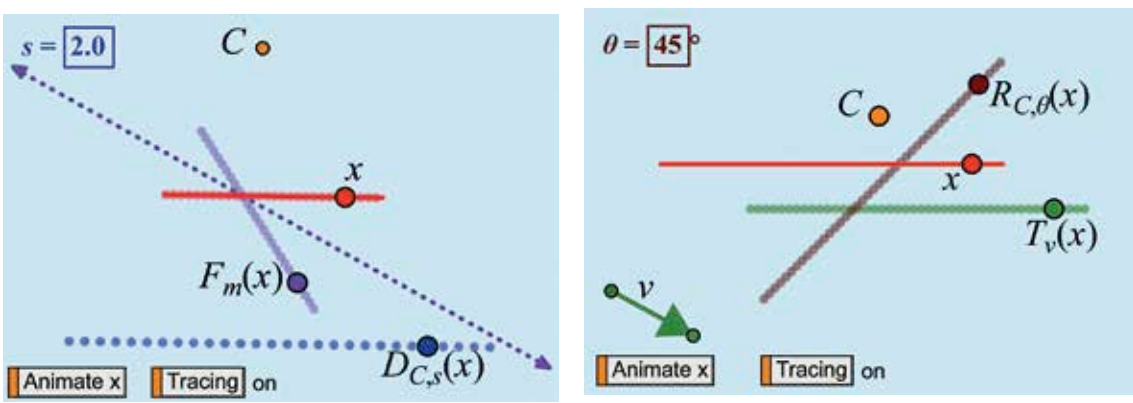
Once students have been introduced to geometric functions and have used activities such as this dilation activity to identify and describe function characteristics, including domain, range, and relative rate of change, they are ready for a sequence of "Cartesian connection" activities that relate geometric functions to symbolically expressed algebraic functions, graphed on Cartesian axes.

Because the algebraic functions with which students are already familiar have one-dimensional numeric values as their input and output, students begin this process with what could be described as a journey from Flatland to Lineland, similar to that of Edwin Abbott's narrator A Square in the classic book *Flatland: A Romance in Many Dimensions* (Abbott 1884). To undertake this journey, students restrict the variables of geometric functions, points in a two-dimensional plane (Flatland), to a one-dimensional domain (Lineland).

Domain, Function, and Range

Restricting the domain in this way is at the heart of the first connection activity, Reduce the Dimension. Students construct examples of four geometric function families (reflection, translation, rotation, and dilation) and label the dependent variables meaningfully. Students move points to manipulate the input value. They observe the dependent variable, as the output value is traced, to discern the behavior of all four functions. (**Fig. 3a** shows reflection and dilation; **fig. 3b** shows rotation and translation.)

Students construct each transformation a second time, this time restricting the independent variable to a segment (shown in red in **figs. 4a** and **4b**). They again vary x to observe the traces and notice that the range of each function is also a segment, though possibly having a different direction or length. (This is one of many rich opportunities for students to think about and discuss relative rate of



(a) (b)

Fig. 4 As x is dragged along its domain (red), each dependent variable traces its range.

change and consider how the relative speed and the direction of the variables determine the direction and length of the range.)

Students may be puzzled at this stage: What would it mean for one of these functions to "live" in Lineland? Can both the domain and range lie in the same one-dimensional space? Even though the range, like the domain, is a segment, the two segments differ in location, direction, or length. So the question becomes, How might each of these functions be adjusted to make the range coincide with the domain?

Small groups of students move mirrors, center points, vectors, and angles to experiment with this question for all four functions. As students describe their results, they may make observations such as these:

- "When we put the dilation center point on the domain line, the range was also on the domain line, no matter what scale factor we chose."
- "We tried the same thing with the rotation center point, but there were only two angles of rotation that worked—180° and 360°."
- "For reflection, we thought the only way to arrange the mirror was perpendicular to the domain line, but when we tried a mirror parallel to the domain line we realized that it worked if the mirror was right on top of the domain line. A funny thing happened: Those two arrangements of the mirror gave exactly the same results as the 180° and 360° rotations!"
- "For translation, we had to make the vector parallel to the domain line, but once we did so, we could make the vector any length we wanted to."

Quantifying the Transformation

In the second activity, Number the Domain, students extend these discoveries by using a number line as the restricted domain, allowing them to easily measure the numeric values of the input and output. As students drag the independent variable (preimage) back and forth along its number line, they observe the resulting continuous variation not only of the output point (image) but also of the numeric values corresponding to both input and output. By attending to these values, students may be surprised—and even excited—to discover that the dilation function multiplies the value of its input by the scale factor and that the translation function adds to the value of its input the directed length of the translation vector. By experimenting with different vectors and scale factors, students realize that these two geometric transformations allow them to perform multiplication and addition on the numeric value of the independent variable.

Students measured each variable to help them relate the numeric changes to the visual changes.

Composition of Functions

In the third activity, Compose on a Line, students compose dilation and translation on the number line. They construct a number line and a dilation function with its domain restricted to the line, using the origin of the number line as the center of dilation. They then construct a translation function using as its input the output of the dilation function. (See **figs. 5a** and **5b**.)

After measuring the values of the independent, intermediate, and dependent variables, students change the dilation scale factor and the length of the translation vector and then drag x to observe the effects on the geometric and numeric representations of the output behavior.

Because all three variables (independent, intermediate, and dependent) in this activity appear on the same number line, their overlapping labels and motions may be confusing. In the next two activities, students build and explore clearer visual representations by separating the input and output axes.

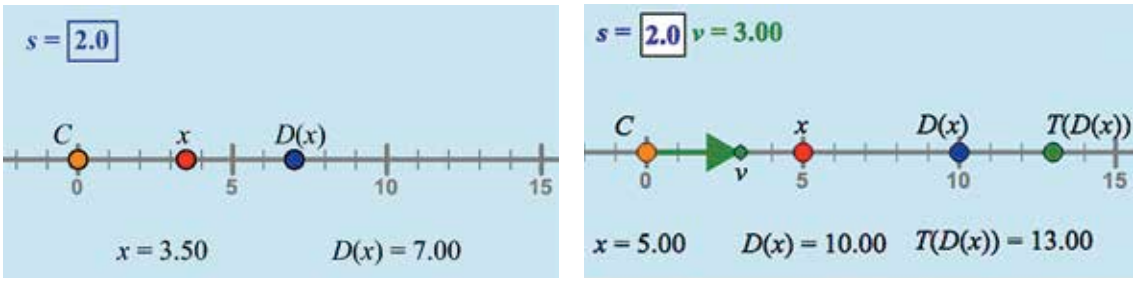
DISTINGUISHING THE OUTPUT

Goldenberg, Lewis, and O'Keefe (1992) invented *dynagraphs* to emphasize function behavior, particularly the relative rate of change of the independent and dependent variables. A dynagraph has two parallel horizontal axes—an input axis above and an output axis below. By putting the variables on parallel axes and connecting them with a segment, students can easily compare their relative motions.

Dynagraphs

In the fourth activity, Create a Dynagraph, students construct a composed dilation-translation function in the form of a dynagraph. They construct the dilation on the upper (input) axis, transfer the resulting intermediate variable to the lower (output) axis, and then construct the translation.

Figure 6a shows the completed construction, in readiness for students to drag, observe, and analyze. (After all the care that students took in the first three activities to get the input and output on the same line, they may be concerned that the variables no longer live on a single line. This concern can lead to an interesting discussion relating to the distinction between an abstract concept and the various representations that we might use to emphasize different features and different



(a) (b)

Fig. 5 Students dilate x by scale factor $s = 2$ (**a**) and then translate the result by a vector v of length 3 (**b**). (Function names have been shortened for readability.)

behaviors.)

Students have measured each variable to help them relate the numeric changes to the visual changes in their locations. They have also measured v, the location of the tip of the translation vector, and they can relate both the scale factor and the translation vector to the displayed values of the variables. In **figure 6b**, students have constructed and traced a segment that connects the input and output variables. Dragging x and analyzing the traces can give students additional insights into function behavior. For instance, a student may notice that one of the traced segments in **figure 6b** connects the origin of the input axis to the tip of the translation vector on the output axis; he or she may try different scale factors and translation vectors to find that this is true for any similar composition (or linear function) and be spurred to explain his discovery.

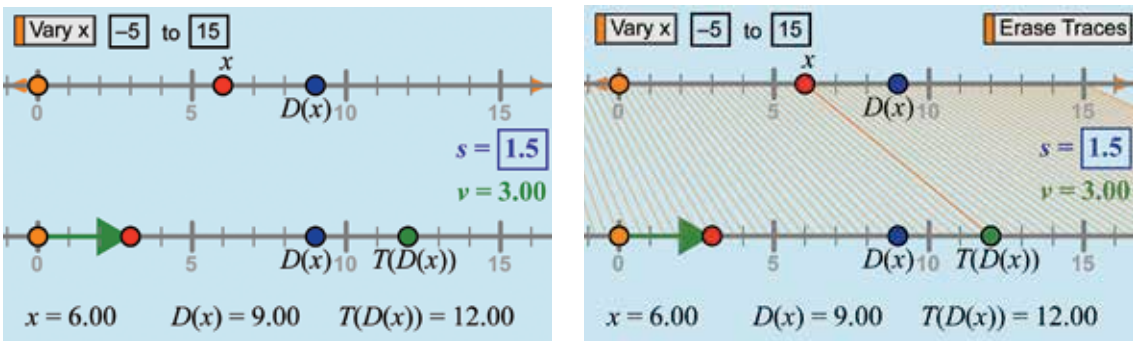
Students are encouraged to act out the motion of the variables by means of finger and hand gestures and also to work as teams to perform a dynagraph dance. In a dynagraph dance, the team uses tape to mark parallel axes on the classroom floor, and team members then take on the roles of variables moving along the axes according to particular scale factors and translation vectors.

Playing the role of dynagraph variables (for instance, for a scale factor of –2 and a vector length of +4) is not only fun for students but also useful in making function behavior concrete and comprehensible. Students should particularly be asked to act out functions with scale factors of 1, 0, and –1 and to explain the observed function behavior in terms of their physical experience.

Students conclude this activity by solving "mystery" dynagraphs that show a pair of connected independent and dependent variables. The students' job is to determine the exact scale factor and translation vector used to produce each mystery function.

A Cartesian Representation

In the fifth activity, Connect to Cartesian, students adopt a different strategy to separate the input and output axes, using a 90-degree rotation rather than a downward translation. As with the dynagraph representation, they begin by constructing a



(a) (b)

Fig. 6 Animating x along the top axis (**a**) leaves traces (**b**) that reveal the behavior of the function. (Function names have been shortened for readability.)

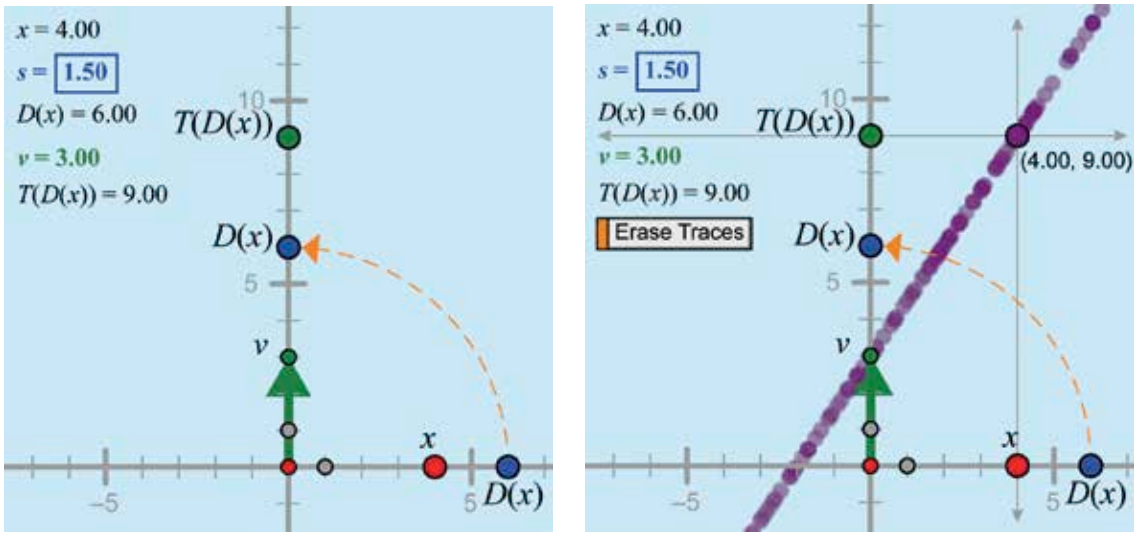


Fig. 7 The dashed orange line shows how the value of $D(x)$ was transferred from the input axis to the output axis (**a**). Subsequent vertical translation (**b**) gives an image that can be used to represent the transformation as a linear function.

dilation on a horizontal number line (the input axis). They then construct a second number line (the output axis) using the same origin, rotate the new line to make it vertical, and construct the translation along this second line.

Figure 7a shows the completed construction, in readiness for varying x.

In **figure 7b**, students have constructed a vertical line through x and a horizontal line through $T(D(x))$ so that the intersection of these two lines tracks the values of both x and $T(D(x))$. By tracing the intersection point of these two perpendicular lines, students realize that they have used transformations to invent the Cartesian graph representation of the linear function $T(D(x)) = s \cdot x + v$, a function they may already know as $y = mx + b$. This connection supports a geometric interpretation of the slope-intercept form of a linear function: The slope is a scale factor applied to the input variable (and thus corresponds to the relative rate of change of the variables), and the intercept is a vertical translation, shifting the value of the output variable.

Students complete this activity by solving mystery graphs, graphs that show a particular path for the traced intersection point in **figure 7b**. The challenge is to adjust their composed function's scale factor and vector so that dragging x causes the intersection to follow the mystery graph.

TRANSFORMATIONS AS FUNCTIONS

In these five activities, geometric functions provide students an alternative environment for engaging with function concepts. This environment emphasizes variation as students drag the variables back and forth on the screen, enabling them to ground abstract function concepts in sensory-motor experiences. The activities also reveal a deep connection between geometry and algebra with the "same" function created by dilation and translation in one realm and by multiplication and addition in the other. As students move from two dimensions to one dimension and from dynagraphs to Cartesian graphs, they attend to function behavior and relative rate of change; come to see multiplication as scaling on the number line and addition as translation on the number line; and grow increasingly aware of (and comfortable with) the composition of multiplication and addition that defines a linear function.

Editor's note: The activities described in this article use Web Sketchpad®, work with any modern Web browser, and can be accessed from http://geometricfunctions.org/links/connecting-functions.

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