Exploring Exponential Relationships:
The Case of Ms. Culver

Ms. Culver wanted her students to understand that exponential functions grow by equal factors over equal intervals and that in working with the general equation $y = b^x$, the exponent (x) tells you how many times to use the base (b) as a factor. She also wanted students to see the different ways that they could represent and make connections with the function. She selected the Pay It Forward task because it provided a context that would help students in making sense of the situation, it could be modeled in several ways (in, for example, a diagram, a table, a graph, and an equation), and it would challenge students to think and reason.

> In the movie *Pay It Forward*, a student, Trevor, comes up with an idea that he thinks can change the world. He decides to do a good deed for three people, and then each of the three people will do a good deed for three more people and so on. He believes that before long good things will be happening to billions of people. At stage 1 of the process, Trevor completes three good deeds. How does the number of good deeds grow from stage to stage? How many good deeds will be completed at stage 5? Describe a function that will model the Pay It Forward process at *any* stage.

Ms. Culver began the lesson by asking students to find a function that would model the Pay It Forward process by any means necessary. She explained that they could use any of the tools that were available in the classroom—for example, graph paper, chart paper, colored pencils, markers, rulers, and graphing calculators. As students began working in their groups, Ms. Culver walked around the room, stopping at different groups to listen in on their conversations and to ask questions as needed (“How did you get that?” “How do the number of good deeds increase at each stage?” “How do you know?”). When students struggled to figure out what to do, she encouraged them to try to represent what was happening at the first few stages visually and then to look for a pattern to see if they could find a way to predict the way in which the number of deeds would increase in subsequent stages.

As she made her way around the room, Ms. Culver also made note of the strategies that students were using (see reverse side) so she could decide which groups she wanted to present their work. She planned to ask the presenting groups to explain what they had done and why and to answer questions posed by their peers. She decided to have the strategies presented in the following sequence. Group 4 would present first since its diagram accurately modeled the situation and would be accessible to all students. Group 3 would go next because its table summarized numerically what the diagram showed visually and made explicit the stage number, the number of deeds, and the fact that each stage involved multiplying by another 3. Groups 1 and 2 would then present their equations, one after the other. Ms. Culver decided that she would give students five minutes at that point to consider the two equations and decide which one they thought modeled the situation better and why.

Below is an excerpt from the discussion that took place after students in the class discussed the two equations that had been presented in their small groups.

Ms. C.: So who thinks that the equation $y = 3x$ best models the situation? Who thinks that the equation $y = 3^x$ best models the situation?

[*Students give a show of hands in response to each question.*]

Ms. C.: Can someone explain why $y = 3x$ is the best choice? Missy, can you explain how you were thinking about this?

Missy: Well, group 1 said that at every stage there are three times as many deeds as at the one that came before it. That is what my group—group 4—found too when we drew the diagram. So the “$3x$” says that it is three times more.

Ms. C.: Does everyone agree with what Missy in saying? [*Many students shake their head in disagreement.*] Darrell, why do you disagree with Missy?

Darrell: I agree that each stage has three times more good deeds than the previous stage; I just don’t think that $y = 3x$ says that. If x is the stage number, like we said, then the equation

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Written by Margaret Smith (University of Pittsburgh), based on a lesson planned and taught by Michael Betler, a graduate student at the University of Pittsburgh. This activity is intended to support the Guiding Principle for Teaching and Learning in *Principles to Actions: Ensuring Mathematical Success for All* (Reston, Va.: National Council of Teachers of Mathematics 2014).

says that the number of deeds is three times the stage number—not three times the number of deeds in the previous stage. So with $y = 3x$ the number of deeds is only 3 more, not 3 times more.

Ms. C.: Other comments?

Kara: I agree with Darrell: $y = 3x$ works for stage 1, but it doesn't work for the other stages. If we look at the diagram, it shows that stage 2 has 9 good deeds. But if you use the equation, you get 6, not 9. So it can't be right.

Chris: Yeah, and $y = 3x$ is linear. If this function were linear, then the first stage would be 3, the next stage would be 6, and then the next stage would be 9. This function can't be linear—it gets really big fast. There isn't a constant rate of change.

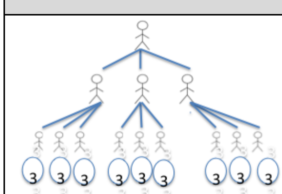
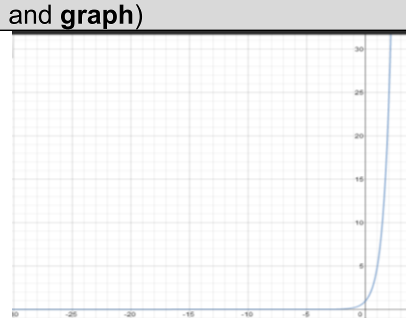
Ms. C.: So let's take another look at group 3's work. Does the middle column help explain what is going on? Devon?

Devon: Yeah. They show that each stage has three times more deeds that the previous one. For each stage, there is one more 3 that gets multiplied. That makes the new one three times more than the previous one.

Angela: So that is why I think $y = 3^x$ best models the situation. Stage 1 had 3 good deeds, stage 2 people had 3 each doing 3 deeds so that is 3^2, stage 3 had 9 people (3^2) each doing 3 good deeds, so that is 3^3. The x tells how many 3s are being multiplied. So as the stage number increases by 1, the number of deeds gets three times larger.

Ms. C.: If we keep multiplying by another 3 like Angela described, it is going to get big really fast like Chris said. Chris also said it couldn't be linear, so take a minute and think about what the graph would look like.

At this point, Ms. Culver asked group 5 to share its graph and proceeded to engage the class in a discussion of what the domain of the function should be, given the context of the problem. The lesson concluded with Ms. Culver telling the students that the function they had created was called *exponential* and explaining that exponential functions are written in the form $y = b^x$. She told students that in the five minutes remaining in the class, she wanted them to explain individually in writing how the equation related to the diagram, the table, the graph and the problem context. She thought that this would give her some insight into what students understood about exponential functions and the relationships among different representations of the function.

<table>
<tr><td>Group 1
(equation, incorrect)</td><td>Group 2
(table, as shown for groups 6 & 7, and equation)</td><td>Group 3
(diagram, as shown for group 4, and table)</td></tr>
<tr><td>$y = 3x$
At every stage there are three times as many good deeds as there were in the previous stage.</td><td>$y = 3^x$</td><td><table>
<tr><td>x
(stages)</td><td></td><td>y
(deeds)</td></tr>
<tr><td>1</td><td>3</td><td>3</td></tr>
<tr><td>2</td><td>3 x 3</td><td>9</td></tr>
<tr><td>3</td><td>3 x 3 x 3</td><td>27</td></tr>
<tr><td>4</td><td>3 x 3 x 3 x 3</td><td>81</td></tr>
<tr><td>5</td><td>3 x 3 x 3 x 3 x 3</td><td>243</td></tr>
</table></td></tr>
<tr><td>Group 4
(diagram)</td><td>Group 5
(table, as shown for groups 6 & 7, and graph)</td><td>Groups 6 and 7
(table)</td></tr>
<tr><td>
So the next stage will be 3 times the numbe there in the current stage so 27 x 3. It is too many to draw. You keep multiplying by 3.</td><td></td><td><table>
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