Exploring the Sine Function: The Case of Barbara Lynch[1]

<u>The Context</u>

Barbara Lynch is a 25–year teacher who has spent her entire career teaching mathematics at Woodlake High School. Barbara's tenth and eleventh-grade Algebra II students have completed units in which they explored how changes in parameters (e.g., m and b in $y = mx + b$) affect the graphs of functions (including linear, absolute value, quadratic, and square root functions, all of which are referred to as parent functions) using graphing calculators, tables, and symbolic representations.

Barbara's class has also been studying trigonometry, specifically working with the sine, cosine and tangent functions. The students made connections to right triangles, worked on application problems, drew angles on the coordinate plane in standard position, defined sine and cosine in terms of x and y coordinates, considered how to describe clockwise and counterclockwise rotations (Barbara told them about the convention of positive and negative angle measures), and solved simple trigonometric equations (e.g., $\cos x = .66$ in right triangles).

In the lessons immediately preceding the one depicted in this case, Barbara's students studied the graphs of $y = \sin x$ and $y = \cos x$. Through observations and discussion, the following conclusions were reached by the class:

1. the graphs of $y = \sin x$ and $y = \cos x$ are repeating the same shape over and over (the student discussion led to them creating and formalizing the definition of periodic);
2. there is a maximum value of y = 1 and a minimum value of y = -1 for both graphs;
3. there is a line halfway between the maximum and minimum values that is situated at y = 0 (the students named this "the midline") ;
4. there is a set distance from the midline to the maximum or minimum value, which the students were told, is called the amplitude.

<u>The Class</u>

As the students entered the room, Barbara had them sit in their pre-assigned small groups. She then asked the students to review the homework assignment on transformations (shown in Figure 1). She circulated around the room as students shared their work with their peers. She noticed that students were able to create functions that reflected over the x-axis and that they were able to shift a function vertically without any difficulty. She also learned that there was some confusion about the affect of a parameter such as a in $y = af(x)$ and with horizontal translations. She made a note to include more investigations about those two transformations in future lessons.

Figure 1. Homework Assignment.

1) Draw any function you want on your graph paper. Call the function $g(x)$
    a. Graph the function $h(x) = g(x) + 3$
    b. Graph the function $j(x) = 2g(x)$
    c. Graph the function $k(x) = g(x+1)$
    d. Graph the function $m(x) = -g(x)$

2) $f(x)$ is a function on the coordinate plane.
    a. Write a function, $a(x)$ that translates $f(x)$ right three units
    b. Write a function $b(x)$ that translates $f(x)$ down six units
    c. Write a function $c(x)$ that reflects $f(x)$ over the y-axis
    d. Write a function $d(x)$ that contracts $f(x)$ by a scale factor of ½

After a few minutes, Barbara called the class together, thanked them for their hard work, and asked them to consider what they had just discussed in terms of the graphs of $y = \sin x$ and $y = \cos x$. She distributed

[1] This case, written by Frederick Dillon, is based on his 35 years of experience teaching high school mathematics.

the task, “Investigate the Graphs of Sine Waves” (shown in Figure 2). The class read the investigation and took a few moments to consider it. By Barbara’s established classroom procedure, the students asked questions they had about the instructions and task before starting to work.

Figure 2. Investigate the Graphs of Sine Waves.

Investigation: For each of the forms of sine functions below, you will explore the graph, its location on the coordinate plane, and how the new graphs are related to the graph of the parent function.

$y = a\sin x$ $\qquad$ $y = \sin x + c$ $\qquad$ $y = a\sin x + c$

Before you start, individually predict how you think the parameters will affect the graphs. Test you predictions using your graphing calculator. Substitute different values for *a* and/or *c*. Use a variety of values including ones that are greater than 1, between 0 and 1, positive and negative. Record your observations.

Next, you will share your observations and work on this with your group.

As a group, use your graphs to answer these questions:

a) Were your predictions correct? State clearly what you expected and what your experiments showed you.

b) How do changes in the values of *a* and *c* affect the parent function $y = \sin x$?

With the class still arranged in small groups, Barbara told them they would have 15 minutes to work, after which there would be a whole class discussion about their findings. As students worked on the task, Barbara walked around the classroom first making sure that everyone was getting started without any difficulty and then monitoring the progress of each group.

Group 1: Barbara stopped at Group 1 as students were discussing their predictions and results for $y = a\sin x$. (figure 3) Chris was commenting that the graphs of the sine waves kept getting taller. Barbara asked, “What do you mean by ‘taller’?” Chris explained that when *a* became larger, the curve became narrower, or more stretched, similar to what the class had noticed with $y = ax^2$. Barbara asked others in the group, “What do you think? Did you get the same results as Chris?” Alex replied, “Yeah, I noticed that the max and min got further from the axis.” Pat added, “Yep, that’s what I found, too. Let’s go on to the next one.” “Whoa, wait a minute,” Barbara commented as she was looking at the group’s recording sheet. “I have a couple of questions. Let’s look at the values you used for *a*.” Barbara continued, “Hmm, you all agreed the amplitude became larger as *a* got larger. So I’m wondering, what’s the smallest *a* value you could use?” She did not specifically remind the students of their previous work with transformations and using different parameter values. Chris commented that two was a good initial value for *a*, since for $y = \sin x$ the coefficient of sinx, that is *a*, is one. Barbara then decided to ask Jordan, a student in the group who had been quiet, about the task. Jordan responded, “Well, I guess *a* could be a negative number. We found a pattern when multiplying by a negative with other function families.” Pat stated that they should start with -2 and see what happened.” Barbara replied, “What do you think will happen to the graph when you use negative values for *a*?” All but one of the students said the graph would get taller and flip, (the other student thought the graph would move down). Barbara told them to test their ideas and said, “Are there other values of *a* you could try also? How do you think you could make the amplitude shorter?” as she moved on to the next group.

Figure 3

Form	Value of a	Value of c	Equation graphed
y = sinx			
	2	0	y = 2sinx
	5	0	y = 5sinx
	3	0	y = 3sinx
	10	0	y = 10sinx

Group 4:
Barbara noticed that group 4 had completed the chart in figure 4. Barbara asked Shayla, “What does your group mean by ‘‘flipped over’?” Shayla explained that ‘flipped over’ meant that the graph reversed over the x-axis. Barbara said, “Let’s talk about the ‘flipped over’ thing. Shayla said the graph is ‘reversed’. What does that mean to the rest of you?” D’Juan recalled “reflection over the *x*-axis” from their work with absolute value and parabolic functions discussion on the previous night’s assignment. Barbara responded, “So, when and how does the sine wave get reflected? Look at your chart. What do you think causes the sine wave to be reflected?”

Figure 4

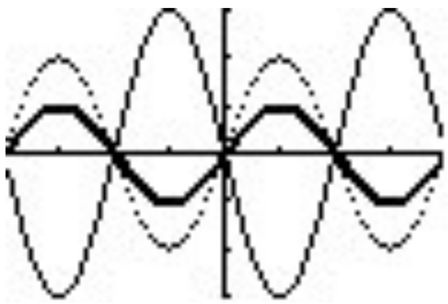
Form	a value	c value	Equation	Effect on shape of graph
y = sinx		0		
	2	0	y = 2sinx	Amplitude =2
	-5	0	y = -2sinx	Amp = 2 and flipped
	-3	0	y = -3sinx	Amp = 3 and flipped
	1.5	0	y = 1.5sinx	Amp = 1.5

Cristel answered, “It seems like it flips over, I mean gets reflected, when *a* is negative.” Barbara asked, “What do the rest of you think about Cristel’s conjecture?” Taylor said, “This is just like our other functions.” The group agrees. Barbara said, “I have another question. You mentioned the amplitude, but you were not specific. Where is the amplitude? You noted the amplitude getting larger. Can you make a smaller amplitude? If so, how?”

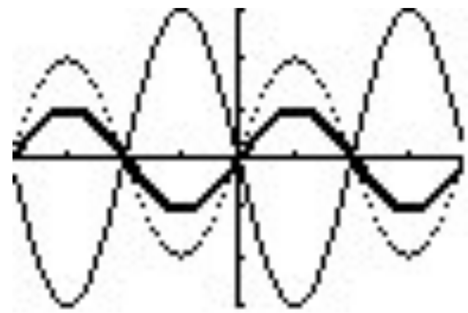
Barbara continued circulating around the room stopping to ask each group questions about their graphs. She often had to prompt students to try positive and negative values for *a* and to test values between -1 and 1. She made notes to herself as she visited each group to remind herself to address specific ideas during the whole-group discussion. Even though students had not gotten to question 3 of the investigation (equations of the form y = *a*sinx + *c*), Barbara noticed that students could easily see and describe the effect *c* had on the graph of the sine wave. Barbara decided it was time to pull the class together for a discussion.

The Discussion

It had become a norm in Barbara's class, after many struggles and much persistence, for students to share and discuss their work publicly. When errors were made, Barbara referred to them as "learning opportunities" and stressed that students often learned more about mathematics by discussing the errors than from reviewing correct solutions. In fact, Barbara modeled this by encouraging her students to point out "learning opportunities" she might make during a lesson.



y = sinx, y =2sinx, y = -3sinx
window [-2π. 2π] by [-3,3}

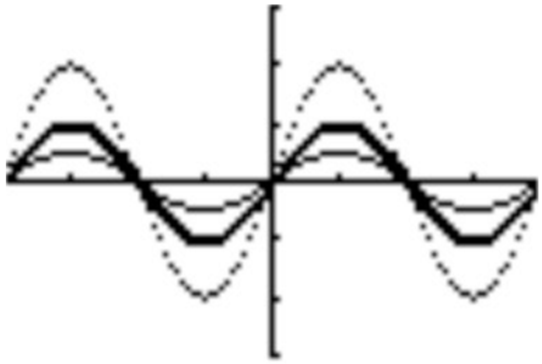


y = sinx, y =4sinx, y = -10sinx
window [-2π. 2π] by [-12,12}

Barbara began the whole-group discussion by asking group one to share and explain their group's graphs (shown below). Howie explained that they graphed y = 2sin$x$, y = -3sin$x$, y = 4sin$x$ and y = -10sin$x$ and found that the larger they made a, the larger the amplitude became, saying, "The amplitude is the same as a." Tamara added that when a values were negative, the sine wave was flipped, or reflected over the x-axis. Barbara asked them where the amplitude was. Tamara pointed to the distance from the midline to 3 on y = -3sinx saying, "That is the amplitude of 3."

Next, Barbara asked the class about Howie's statement and wrote on the board under a heading "WHAT WE DISCOVERED", "The amplitude is a," and, "A negative a value causes the sine function to reflect over the x-axis." Barbara then asked the class, "Do you agree with Howie and Tamara's group?", to which the majority of the students nodded. After a few moments of silence, Pat mentioned that the negative sign only caused the graph to flip, but that a distance (the amplitude) could not be negative. Howie immediately said, "You know what I meant," so Barbara asked him to correct the statement. He pondered for a few seconds and said, "the amplitude is the positive of a." Pat then asked, "Isn't it really the absolute value?" A brief discussion on definitions and terminology led to the class agreeing to use "The amplitude is equal to the absolute value of a." Barbara asked, "How does this fit your predictions? Does the amplitude always get larger?"

She then asked group 6 to share their graphs. Reneé showed their graphs of y = sinx (dark) y = 2sinx, y = ½ sinx in the first picture and y = sinx (dark) y = -¼ sinx and y = -3sinx (second picture). "We used the same viewing window for all the graphs. We think that when a is a fraction, the graph gets a smaller amplitude," she explained.



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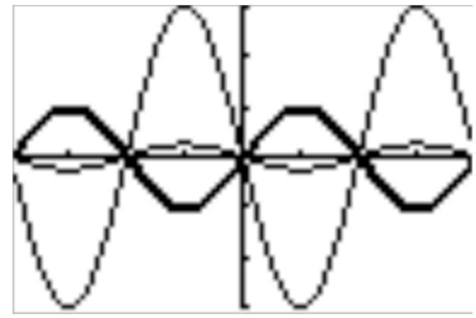
window [-2π. 2π] by [-3,3}



y = sinx, y =-1/4sinx, y = -3sinx
window [-2π. 2π] by [-3,3}

See when *a* is ½ and -¼ the amplitude is less than 1. It is the absolute value of *a*." Barbara asked the class, "Do you agree with Reneé?" Students were nodding, so Barbara continued, "They said when *a* is a fraction the amplitude shrinks. Is that always true?" Taylor, from another group, remarked that he also used a value of 1/3 for *a* and that the sine wave got "shorter," so he thought it was probably always true. "Anyone else want to comment?" Barbara asked. Since no one volunteered to respond, Barbara continued. "Something is confusing me. You agreed with Howie and Tamara that as the *a* value got larger, the amplitude got larger, and you agreed with Reneé and Taylor that when *a* is a fraction, the graph gets "shorter". What would happen if *a* had a value of, say, 2 ½? Would the amplitude be larger or smaller than the graph of $y = \sin x$?" Many students answered that the amplitude would be larger, and Barbara asked them to explain their thinking. "Because 2½ is more than 2 and we know the sine wave gets a larger amplitude the larger *a* is," Shayla answered. "But *a* is a fraction", Barbara responded. "I thought you said the amplitude got smaller when *a* was a fraction." Many students responded that they meant fractions that were less than 1. "What about -3 ¼?" Barbara inquired. "It's less than 1." "You know what we mean," the students shouted. "You're always so picky!" Barbara continued with the discussion, prompting the students to use precise language and notation when answering questions. She made certain students discussed the fact that values of *a* between -1 and 1 resulted in a sine wave that was contracted – that is, had a smaller amplitude, and that values of *a* greater than 1 or less than -1 resulted in dilated (or stretched) sine waves. She also asked students to represent the relationship symbolically and added to the list of properties, "If $-1 < a < 1$, the sine wave has a smaller amplitude than $y = \sin x$ and if $a > 1$ or $a < -1$, the sine wave has a larger amplitude than y" and drew rough sketches of sine waves to represent each case. Several students then noted that the amplitude of $y = \sin x$ was one, and that the amplitude of $y = a\sin x$ would then be *a because* each y-value of the function was multiplied by *a*.

Since there were only a few minutes left in the class period, Barbara asked the students to begin the discussion about the effect of "+ *c* " in their small groups and told them they would continue the discussion tomorrow. She also asked the students to think about the equations of the form $y = a\sin x + c$ and to be prepared to discuss their conclusions about the effects the changes in the parameters on both a sine wave cosine wave tomorrow.