MELISSA D. BOSTON, MARGARET S. SMITH, AND AMY F. HILLEN

Building on Students' Intuitive Strategies to Make Sense of Cross Multiplication

MIDDLE-GRADES STUDENTS' UNDERSTANDING of proportional relationships should be fostered through problem solving and reasoning (NCTM 2000). Toward this end, instruction in proportionality should expose students to a variety of strategies and allow students to gain experience modeling proportional situations (Langrall and Swafford 2000). Students should be given ample opportunities to develop intuitive strategies based on factor-of-change ("how many times as many") relationships (Cramer and Post 1993). Research has shown that middle-grades students are more successful at solving proportion problems when applying strategies that allow them to reason about proportional relationships instead of using a cross-multiplication approach alone (Ben-Chaim et al. 1998; Cramer and Post 1993). This research suggests that introducing the cross-multiplication procedure "should be delayed until after students have had an opportunity to build on their informal knowledge and develop an understanding of the essential components of proportional reasoning" (Langrall and Swafford 2000, p. 261). Once solid conceptions of proportionality have been developed, cross multiplication can be introduced as an efficient algorithm for solving any missing-value proportion problems—especially those for which the numbers in the problem make intuitive strategies difficult or cumbersome to apply.

In this article, we explore how problems that promote students' understanding of factor-of-change relationships can also be used as a basis for making sense of cross multiplication. Through this approach, students maintain a focus on meaning and understanding and avoid the "unfortunate side effects [that can occur] when students do not adequately understand when the [cross multiplication]

MARGARET SMITH *is an associate professor at the University of Pittsburgh and advises graduate-student researchers* MELISSA BOSTON *and* AMY HILLEN.

The information in this article is based on work supported by the National Science Foundation (NSF) under the auspices of the COMET project (Cases of Mathematics Instruction to Enhance Teaching), grant no. 9731428. Any opinions, findings, conclusions, or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of the NSF.

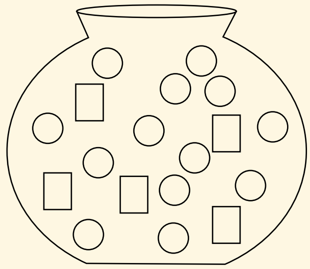
method is appropriate to use" (NCTM 2000, p. 221). We begin our discussion by focusing on the events that unfold in Marie Hanson's sixth-grade classroom during a lesson on understanding ratios and proportions (Smith et al. forthcoming), and use this lesson as a context for considering how factor-of-change relationships might be used to assist students in understanding why cross multiplication works.

Marie Hanson's Lesson

MARIE HANSON ENCOURAGED HER STUDENTS TO develop a variety of strategies for reasoning about proportional relationships. Ms. Hanson wanted students to make sense of problems that demanded thinking about relative change and sought to provide opportunities for students to think informally, but systematically, about quantities and their relationships. She believed that a repertoire of strategies would give students flexibility in working with different contexts and numbers in proportional situations.

Ms. Hanson and her colleagues had decided to weave ratio and proportion problems throughout the sixth-grade curriculum rather than isolate these ideas in a single unit at the end of the year, which was the approach taken by the textbook. In late fall, Ms. Hanson presented her students with the Candy Jar problems shown in **figure 1.**

Ms. Hanson sequenced the three Candy Jar problems in a way that built on students' prior knowledge but encouraged them to consider new ways of thinking about proportional relationships. In previous lessons, Ms. Hanson's students had encountered problems similar to the one presented in question 1 but with ratios that could be reduced to unit-rates composed of whole numbers (i.e., 3 lollipops for each child or 6 Twinkies in each package). Ms. Hanson's students solved these problems by circling combinations of objects (e.g., 3 lollipops and 1 child), which indicated an intuitive sense of unit-rate relationships. Students then generated equivalent ratios through the use of repeated addition. Ms. Hanson next wanted students to analyze factor-of-change relationships and posed a situation in the Candy Jar problems in which the unit-rate relationship was not convenient to reason from (i.e., 2.6 jawbreakers for each Jolly Rancher does not make sense in a real-world context) and was not easy to manipulate mathemati-



This candy jar contains Jolly Ranchers (the rectangles) and jawbreakers (the circles).

1. What is the ratio of Jolly Ranchers to jawbreakers in the candy jar? Write as many ratios as you can that are equivalent to the first ratio that you wrote.

2. Suppose you had a new candy jar with the same ratio of Jolly Ranchers to jawbreakers, but it contained 100 Jolly Ranchers. How many jawbreakers would you have?

3. Suppose you had a candy jar with the same ratio of Jolly Ranchers to jawbreakers, but it contained 720 candies. How many of each kind of candy would you have?

Fig. 1 The Candy Jar problems

Jolly Ranchers	Jawbreakers
5	13
+5	+13
10	26
+5	+13
15	39
20	52

(a)
Creating the table of values through repeated addition

	Jolly Ranchers	Jawbreakers	
	5	13	
	10	26	
×2	15	39	×2
	20	52	

(b)
Using the table of values to identify the factor-of-change relationship between row 2 and row 4

Jolly Ranchers	Jawbreakers
1	2.6
×5	×5
5	13
10	26
15	39

(c)
Using the table of values to identify the factor-of-change relationship between 1:2.6 and 5:13

Fig. 2 The table of values for the Candy Jar problems

cally (i.e., students' previous strategy of forming equivalent ratios by repeatedly adding the unit-rate would be difficult to apply). When Ms. Hanson asked her students what made the Candy Jar problem more difficult than earlier problems, Jerry responded, "You can't get back to one, like we were able to do with the lollipops and the Twinkies. Remember how we circled 3 lollipops for each 1 of the children?" Then he paused for a moment as if a light bulb had just gone on in his head. "Well, I guess you could, but then you'd have little teeny pieces of candy" (Smith et al. forthcoming). Jerry's comment acknowledged that determining a unit rate was not a useful way of thinking about the Candy Jar problems in terms of both the context and the mathematics.

By choosing numbers that made the unit-rate relationship difficult to work with, Ms. Hanson intended to encourage students to use factor-of-change relationships to generate equivalent ratios. However, students could approach question 1 by repeatedly adding 13 more jawbreakers and 5 more Jolly Ranchers. One pair of students in Ms. Hanson's classroom, Jerlyn and Kamiko, created the table of values in **figure 2a** in this way. To move students beyond additive strategies, Ms. Hanson asked students to observe possible relationships between different rows in the table. As illustrated in **figure 2b,** one student noticed that "you can get from the second row to the fourth row by multiplying by 2. Two times 10 equals 20 and 2 times 26 equals 52" (Smith et al. forthcoming). Students also discussed whether the ratio of 2.6 jawbreakers to 1 Jolly Rancher (identified earlier in the lesson as the unit-rate) would "fit" into the table. Students decided that it should go into the first row "because the numbers were the smallest" and that multiplying each term by 5 would generate the ratio of 5:13 in the next row of the table (see **fig. 2c**). Identifying this multiplicative relationship convinced students that 1:2.6 "fit" into the table and was equivalent to 5:13 and to the other ra-

Jerry's strategy

$$1 \text{ J.R.} \xrightarrow{\times 100} 100 \text{ J.R.}$$
$$2.6 \text{ J.B.} \xrightarrow[\times 100]{} 260 \text{ J.B.}$$

Owen's strategy

$$5 \text{ J.R.} \xrightarrow{\times 20} 100 \text{ J.R.}$$
$$13 \text{ J.B.} \xrightarrow[\times 20]{} 260 \text{ J.B.}$$

Fig. 3 Strategies for question 2

Danielle's strategy

$$5 \text{ J.R.} \xrightarrow{\times 40} 200 \text{ J.R.}$$
$$18 \text{ Total} \xrightarrow[\times 40]{} 720 \text{ Total}$$

Joshua's strategy

Jolly Ranchers	Jawbreakers	Total
5	13	18
10	26	36
15	39	54
20	52	72
25	65	90
…	…	…
50	130	180
200	520	720

$\times 4$ $\times 4$

Fig. 4 Strategies for question 3

tios, as well. These exchanges served to highlight multiplicative relationships and to provide students with another way of reasoning about equivalent ratios.

Ms. Hanson also prompted students to reason with factor-of-change relationships by incorporating large numbers into questions 2 and 3. In contrast to question 1, repeated addition becomes a very tedious strategy to pursue with problems involving large numbers. Some students in Ms. Hanson's classroom continued to extend the table introduced by Jerlyn and Kamiko **(fig. 2a),** whereas others used strategies based directly on factor-of-change relationships, as shown in **figure 3.** For example, Jerry solved question 2 by reasoning, "If one Jolly Rancher turns into 100 Jolly Ranchers, it must have been multiplied by 100. And so the 2.6 jawbreakers also have to be multiplied by 100." Another student, Owen, contended that "I knew it was 260 jawbreakers because you had to multiply the 5 Jolly Ranchers by 20 to get 100, so you'd also have to multiply the 13 jawbreakers by 20 to get 260—kinda like what Jerry did, but I started with 5 and 13, not 1 and 2.6" (Smith et al. forthcoming).

$$\frac{5}{18} = \frac{x}{720}$$
$$18x = 3600$$
$$\frac{18x}{18} = \frac{3600}{18}$$
$$x = 200$$

Fig. 5 Angelica's method

Question 3 also elicited strategies based on factor-of-change relationships as shown in **figure 4.** Danielle multiplied the original number of Jolly Ranchers by 40, since there were 40 times as many total candies in the new jar. Joshua began by creating a table through repeated addition, but soon realized that a more efficient strategy might be possible. When he reached 180 total candies (50 Jolly Ranchers and 130 jawbreakers), he figured out that four candy jars that hold 180 candies would be needed to create a jar with 720 total candies. He realized that he could quickly obtain 720 total candies by multiplying each number of candies by 4.

Problem 3 also elicited an unexpected strategy—the cross-multiplication algorithm. Although Ms. Hanson had planned to introduce this approach at some point during the year, she wanted her students to have experience solving problems using more intuitive strategies first. But when Angelica presented her "*much* easier" way to solve the problem (shown in **fig. 5**), Ms. Hanson needed to decide on the spot how to help her students link these two approaches in a way that made sense and built on their current understandings.

Imagine yourself in Ms. Hanson's position. You have taken the time to help students develop intuitive approaches and want to continue to focus on sense-making. How could you help students make sense of cross multiplication, drawing on their current understanding of proportional relationships? In the next section, we suggest a way of connecting the mathematical ideas underlying cross multiplication with informal strategies based on a factor-of-change relationship.

Connecting Cross Multiplication to the Factor-of-Change Strategy

LET'S CONSIDER WHAT STUDENTS WOULD NEED to understand and how you might design instruction to make the connection between factor-of-change strategies and the cross-multiplication algorithm. First, students would need to recognize that the new ratio generated by applying the factor-of-change is equivalent to the original ratio (see **fig. 6a**). Ms. Hanson's students had discovered that equivalent ratios could be generated by multiplying by a factor-of-change (i.e., a scale factor) when discussing the

$$\frac{5}{18} = \frac{200}{720}$$

(a)
The ratios are equivalent.

$$\frac{5}{18} = \frac{5 \bullet 40}{18 \bullet 40}$$

(b)
The new ratio can be rewritten to expose the scale factor.

$$5 \bullet 18 \bullet 40 = 18 \bullet 5 \bullet 40$$

(c)
The cross products are equal.

Fig. 6 What students need to know to connect the factor-of-change relationship to cross multiplication

relationships between different rows of Jerlyn and Kamiko's table **(fig. 2).** Students then began to use factor-of-change relationships directly to solve question 2 (e.g., Jerry's and Owen's strategies in **fig. 3**), and had even articulated that multiplying both terms of the original ratio by the same amount was necessary to "grow things at the same rate" (Smith et al. forthcoming). In Danielle's strategy, 5:18 was "scaled up" by a factor of 40 to obtain the new ratio—that is, 5 and 18 were both multiplied by 40 to obtain 200:720. Drawing on students' own use of factor-of-change strategies, such as Danielle's work, you can establish that the new ratio of 200:720 is equivalent to the original ratio of 5:18 **(fig. 6a).**

Second, students should understand that the new ratio can be rewritten as the terms of the original ratio multiplied by the scale factor. Having students rewrite 200:720 as 5 • 40:18 • 40 (shown in **fig. 6b**) makes the factor-of-change relationship salient by showing that the initial amounts were *both* multiplied by 40.

Third, students must recognize that each cross product consists of the same three factors (the two terms of the original ratio and the scale factor) and that the ordering of the factors does not matter because of the commutative property of multiplication. Students might be asked to multiply the numerator of one ratio with the denominator of the other (and vice versa) and to compare the two results. Once students observe that the two cross products are equal, ask them to explain why this result occurs. Students should be able to explain that each cross product consists of 5 and 18 (the terms of the original ratio) and the scale factor of 40 (as shown in **fig. 6c**). Students might then be pressed to consider (1) whether the cross products would be equal for any pair of equivalent ratios and (2) why the ratios *must be*

$$\frac{5}{18} = \frac{n}{720}$$ Find the missing value in the proportions.

$$\frac{5}{18} = \frac{n}{18 \bullet 40}$$ Rewrite 720 to expose the scale factor of 40.

$$\frac{5}{18} = \frac{5 \bullet 40}{18 \bullet 40}$$

Factor-of-change approach

$$5 \bullet 18 \bullet 40 = 18 \bullet n$$
$$\frac{5 \bullet 18 \bullet 40}{18} = \frac{18 \bullet n}{18}$$

Cross-multiplication approach. Note that the missing factors are 5 and 40.

5 • 40 = 200 The missing term is 200.

Fig. 7 Comparing factor-of-change with cross multiplication strategies

equivalent for this result to occur. Algebra students may be able to construct a proof symbolically, and younger students may be able to reason from their understanding of the factor-of-change relationships between equivalent ratios. If the two ratios are not equivalent, then the new ratio could not be written as the terms of the original ratio multiplied by a scale factor. The cross-multiplication algorithm would not hold.

Finally, students would need to see how and why cross multiplication is useful when one of the values in the proportion is missing. Since the cross products can be set equal, cross multiplying produces an equation that consists of the same three factors on both sides. Thus, the equation can easily be solved to determine the missing value. Care should be taken at this point not to present the algorithm as a "neat trick." We suggest comparing each step of the cross-multiplication algorithm with a factor-of-change strategy **(fig. 7).** Looking at a factor-of-change strategy (e.g., Danielle's strategy) and the cross-multiplication strategy (e.g., Angelica's strategy) side by side might provide an opportunity to make connections between students' prior understanding of factor-of-change relationships and the steps of the cross-multiplication algorithm. Alternatively, if cross-multiplication does not emerge spontaneously in your classroom, you can use a factor-of-change strategy to initiate a discussion that will allow students to develop the procedure. (Note also that a similar connection to cross multiplication can be made using the constant unit-rate relationship between equivalent ratios. We invite you to construct this argument using equivalent ratios in which the unit-rate relationship is easy to identify and compute.)

Allowing students to test cross multiplication on missing-value proportion problems in which the factor-of-change strategy is easy to recognize and compute might help strengthen students' understanding of factor-of-change relationships. It might also help develop an understanding of why cross multiplication works. Encourage students to use both strategies and compare similarities and merits of each. Of course, eventually posing problems in which the factor-of-change is not easy to recognize or compute (i.e., large numbers or decimals) will be necessary to have students realize when cross multiplication is especially helpful. Indicate to students that later applications in algebra also make this algorithm powerful to know and to *understand*.

Conclusion

GIVING STUDENTS OPPORTUNITIES TO DEVELOP an understanding of factor-of-change relationships not only gives students effective strategies for reasoning about proportional relationships but also provides a foundation on which to build an understanding of why the cross multiplication algorithm works. Rather than apply cross multiplication as a rote procedure, students can make the transition from intuitive strategies to cross multiplication in a way that continues to promote meaning and understanding.

References

Ben-Chaim, David, James T. Fey, William M. Fitzgerald, Catherine Benedetto, and Jane Miller. "Proportional Reasoning among Seventh Grade Students with Different Curricular Experiences." *Educational Studies in Mathematics* 36 (1998): 247–73.

Cramer, Kathleen, and Thomas Post. "Proportional Reasoning." *Mathematics Teacher* 86 (May 1993): 404–7.

Langrall, Cynthia W., and Jane Swafford. "Three Balloons for Two Dollars: Developing Proportional Reasoning." *Mathematics Teaching in the Middle School* 4 (December 2000): 254–61.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

Post, Thomas, Merlyn J. Behr, and Richard Lesh. "Proportionality and the Development of Prealgebra Understandings." In *The Ideas of Algebra, K–12,* 1988 Yearbook of the National Council of Teachers of Mathematics (NCTM), edited by Arthur F. Coxford and Albert P. Shulte, pp. 78–90. Reston, Va.: National Council of Teachers of Mathematics, 1988.

Smith, Margaret S., Edward A. Silver, Mary Kay Stein, Marjorie Henningsen, and Melissa Boston. *Cases of Mathematics Instruction to Enhance Teaching (COMET).* Forthcoming. □