CHAPTER 1

Algebraic Expressions

In school algebra, an algebraic expression is a phrase or term that contains numbers and literal symbols (variables) connected by operations. It represents a calculation that will result in a numerical value whenever each of its variables is replaced by a number and the indicated operations are performed.

The Common Core State Standards for Mathematics (CCSSM) recommend the following main goals for the teaching and learning of algebraic expressions (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center and CCSSO] 2010, p. 62):

- Correct application and a sound understanding of the laws of arithmetic and algebra, as well as respecting the conventions about algebraic notations and the use of parentheses and the order of operations
- Reading expressions with comprehension, which implies the analysis of the underlying structure of an expression, including identifying equivalent expressions despite their different appearance
- Interpreting expressions as models of real-world contexts/situations
- Interpreting parts of an expression (and sometimes the whole expression) as a single entity, and identifying/recognizing structure, which enables applying algebraic laws for the purpose of achieving a simpler expression
- Flexibility of choosing a specific form among several equivalent forms of an expression in order to reveal meaning or explain a property represented by it
- Creating expressions in order to represent (*a*) desired properties, or (*b*) the general aspect of a geometrical pattern, or (*c*) a model of real-world phenomena
- Viewing expressions as relationships (sometimes functional relationships) between an input (an independent variable) and an output (a dependent variable)

The tasks proposed in this chapter were designed to pursue these goals and to support the application and development of the cognitive competencies relevant to each topic. We grouped the CCSSM goals as follows:

Procedural fluency. To correctly "handle" a given algebraic expression includes the following skills:

- Correctly substituting a given number for a variable, and knowing that within the same expression the same value should be substituted in any instance of that variable. Sometimes students should be able to substitute a known value for an expression, even if the values of the individual variables constituting it are not known (see task 1.4)
- Recognizing which kinds of numbers should be substituted in order to obtain a specific result or a set of results with certain characteristics (see task 1.1)
- Applying procedures such as simplification, expansion, and rearrangement of terms to an expression in order to derive an equivalent expression (see task 1.6)
- Considering possible substituted values that yield a given number either by setting up an equation or by other idiosyncratic ways (see task 1.2)
- Being aware of common procedural mistakes, and being able to detect and correct them in one's own or in others' work (see task 1.5)

Creating expressions for a desired goal. Students should be able to create expressions of their own in order to express or represent a desired outcome or to model a situation, as in the following ways:

- When given a substitution value and the evaluated value, being able to construct expressions that fulfill the constraint (see task 1.3)
- Creating expressions that represent generalized patterns of numbers (see task 1.8)
- Integrating the creation of expressions with procedural fluency (see task 1.9)
- Exploring surprising and unexpected number properties and relationships (see task 1.12)

Chapter Tasks

The following table presents the twelve tasks in this chapter with brief descriptions of their content.

Task Name	Description
1.1 The Racing Track: A Substitution Game	Students advance on a game board by analyzing algebraic expressions and substituting numbers that provide a positive result. (Obtaining negative results causes a penalty of having to move backward.)
1.2 Working Backward: From Results to Substituted Numbers	Students are involved in a reverse process: They are given an expression and a set of results or the characteristics of a result, and they must find the corresponding set of substituted numbers.
1.3 Creating Algebraic Expressions	Students are given substituted numbers and the results of their substitution, and they are required to create suitable expressions.
1.4 A Global View of Algebraic Expressions	Students use the given value of an algebraic expression in order to find the value of another expression.
1.5 Take a Quiz / Make a Quiz	Students get involved in the thinking processes related to solving a short multiple-choice test, and they apply their insights by designing a test of their own.
1.6 Express Yourself!	Students create equivalent expressions by themselves under given constraints, and they are confronted with the need to reflect, compare, check, and discuss algebraic procedures and concepts.
1.7 Products of Two Binomials	Students multiply two binomials by constructing products of a given format and then classify them according to the number of terms in their simplified expansion.

1.8 Growing Towers	Students solve a visual pattern-generalization task that follows a sequence of steps consisting of numerical examples → generalization → justification → applications, aimed at providing meaning to the procedure of adding similar terms.
1.9 Dot-to-Dot Arrangements	Students become acquainted with three sequences of dot arrangements, generalize their patterns, connect between methods of dot counting and algebraic expressions, and justify the equivalence of seemingly different expressions by using algebraic procedures.
1.10 Handshakes, Business Cards, and Diagonals	Students analyze, generalize, justify, and apply three similar (but not equivalent) counting contexts: the number of handshakes, the number of business cards exchanged in a meeting, and the number of diagonals in a polygon.
1.11 Expressions of Reversed Numbers	Students use algebra or other alternative methods to investigate (i.e., become acquainted with numerical examples, generalize, justify, and apply) relations between pairs of numbers formed by the same digits written in reversed order.
1.12 Curious Properties of Consecutive Numbers	Students investigate a numerical property of sets of four consecutive numbers, and they are then required to provide an example of a similar property.

Competencies

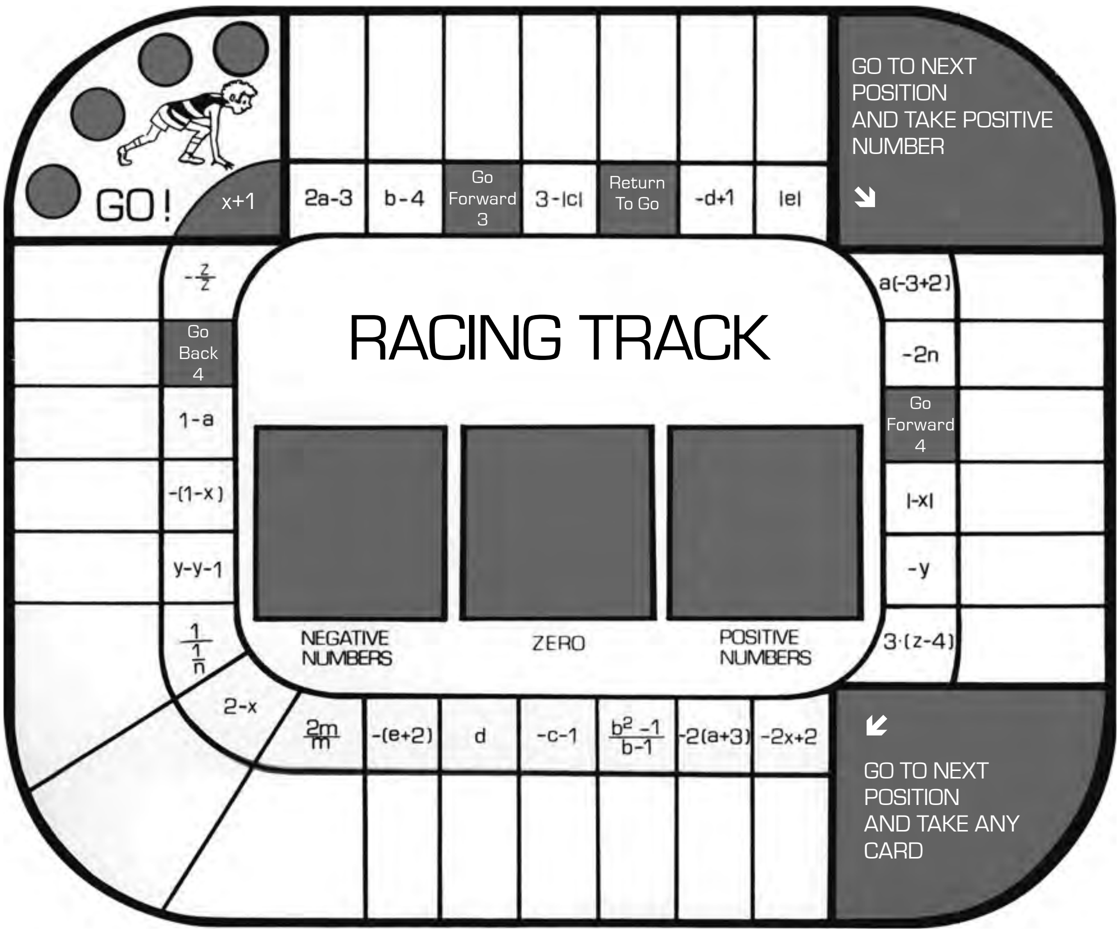
The competencies that are relevant to the tasks in this chapter are summarized in the following table.

Competencies / Tasks	Understanding and applying concepts	Global comprehension	Divergent thinking	Representing, modeling, and interpreting	Reverse thinking	Generating examples	Monitoring one's own or others' work	Generalizing	Justifying and proving
1.1 The Racing Track: A Substitution Game	√				√		√		√
1.2 Working Backward: From Results to Substituted Numbers	√				√	√	√	√	√
1.3 Creating Algebraic Expressions	√		√		√	√	√	√	√
1.4 A Global View of Algebraic Expressions		√		√	√		√		
1.5 Take a Quiz / Make a Quiz	√		√		√		√		
1.6 Express Yourself!	√		√		√		√		
1.7 Products of Two Binomials	√				√	√	√		
1.8 Growing Towers	√	√		√	√		√	√	√
1.9 Dot-to-Dot Arrangements	√	√		√	√		√	√	√
1.10 Handshakes, Business Cards, and Diagonals	√	√		√	√			√	√
1.11 Expressions of Reversed Numbers				√		√	√	√	√
1.12 Curious Properties of Consecutive Numbers	√	√	√		√	√		√	√

Task 1.1

The Racing Track: A Substitution Game

This game consists of advancing runners on the board (the "race track") until one player (out of the two to four playing) completes two full rounds and is declared the winner. The race track consists of consecutive stations, each of which contains a specific algebraic expression (see the game board below).



At the beginning of her turn, the player considers the algebraic expression of her present position and declares whether she plans to substitute in her expression a negative number, a positive number, or zero. If the choice is either a negative or a positive number, the absolute value of the number to be substituted is decided by throwing a die. The result of the substitution determines the number of steps for moving forward or (in the case of a negative result) backward. If the result is undefined (division by zero), the runner has to return to the "Go" space.

Note: The Racing Track game was originally developed at the Department of Science Teaching at the Weizmann Institute of Science, and it is described in detail in Friedlander (1977).

About the Task

Unlike the other activities in this book, this task involves playing a game. The game provides a motivating environment for practicing substitution of numbers in a variety of algebraic expressions. By playing the game, substituting numbers in algebraic expressions becomes a purposeful activity—advancing one's runner as quickly as possible. Players must consider possible results and choose the sign of the substituted number according to the expected or desired outcome. In order to advance their position and to detect possible errors, players are encouraged to monitor both their own and their peers' choices and moves. Students may realize, for example, that in some expressions the following are true:

- It is worthwhile to choose a negative number in order to obtain a positive result (e.g., $-d + 1$).
- Choosing the kind of number may be obvious, but the outcome may still depend (to a certain degree) on luck—for example, $3(z - 4)$.
- The expression is equivalent to a constant number, and thus the decision about the sign and the outcome of the (nonzero) number does not matter (e.g., $\frac{2m}{m}$).

By playing the game, students are led to perform many substitutions and to analyze various types of algebraic expressions.

Comments and Solutions

This game board was designed to include a variety of algebraic expressions, and students are required to make decisions based on their inspection of the expressions. Even after making correct decisions, the outcome of the dice may hold some surprises.

The following are some decisions students may encounter while playing the game:

- For $\dfrac{1}{\frac{1}{n}}$, only a positive number will move the runner forward.
- For $-2n$, only a negative number will move the runner forward.
- For $2 - x$, all negative numbers, zero, and 1 will move the runner forward.
- For $|-z|$, any positive or negative number, except zero, will move the runner forward.
- For $2a - 3$ and $3(z - 4)$, the correct choice is a positive number, but the result still depends on luck.
- For $\frac{b^2 - 1}{b - 1}$, zero is the only sure choice.
- For $y - y - 1$, substituting any number will result in going backward.

Competencies

Understanding and applying concepts

In the process of playing the game, students substitute numbers in expressions, make calculations, analyze expressions, and consider the convenience of certain substitutions, as compared to others.

Reverse thinking

The strategies of the game are based on finding numbers for an optimal move—i.e., their substitution in a given expression provides a larger positive result. Thus, for a given expression and a desired result, the player must look for one or more appropriate substituted numbers.

Monitoring one's own or others' work

At each stage of the game, students analyze their or the other players' expression, predict results, choose a category of numbers according to their prediction, implement their choice (namely, they perform a calculation according to the outcome), and check whether and how their choice fits their expectations.

Justifying and proving

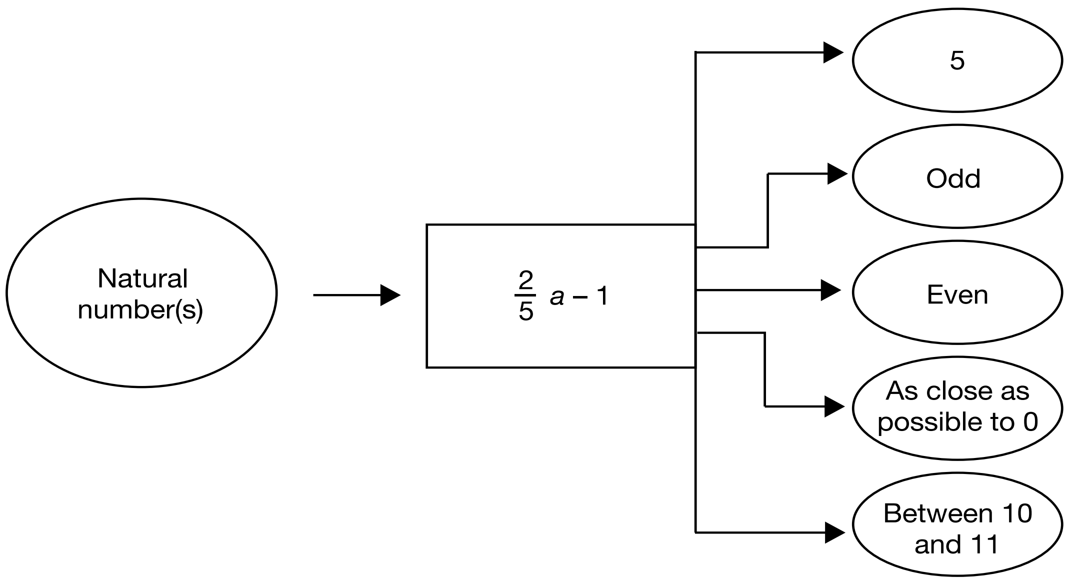
The game allows students to present and justify their reasoning, and eventually to discuss some of their opponents' moves.

Task 1.2

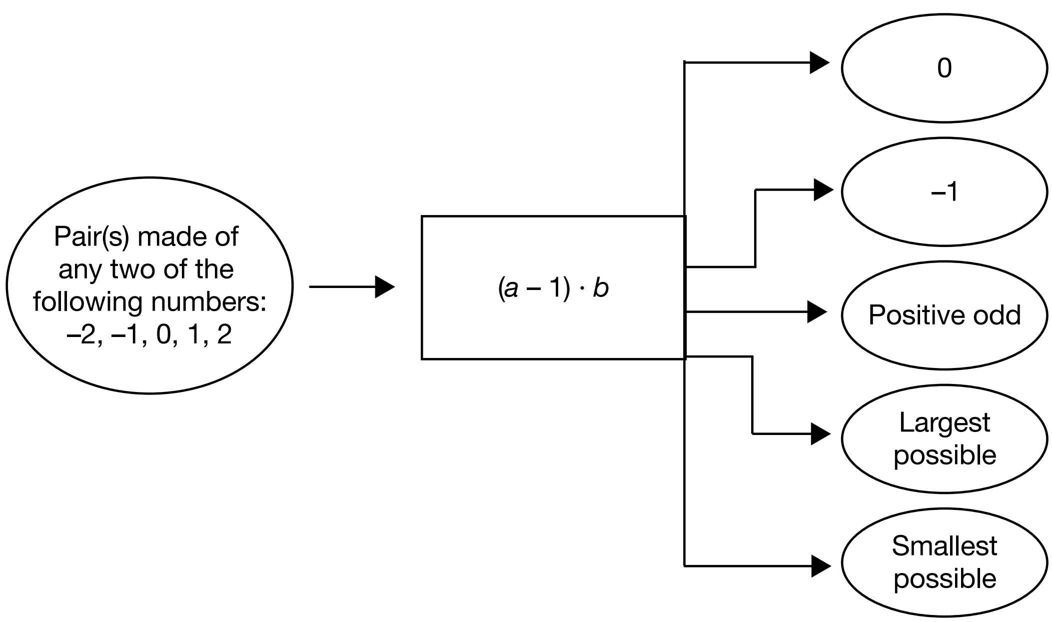
Working Backward: From Results to Substituted Numbers

Five possible results of substitution for two given expressions are provided below. In each case, find all the substitutions that yield these results.

(a)

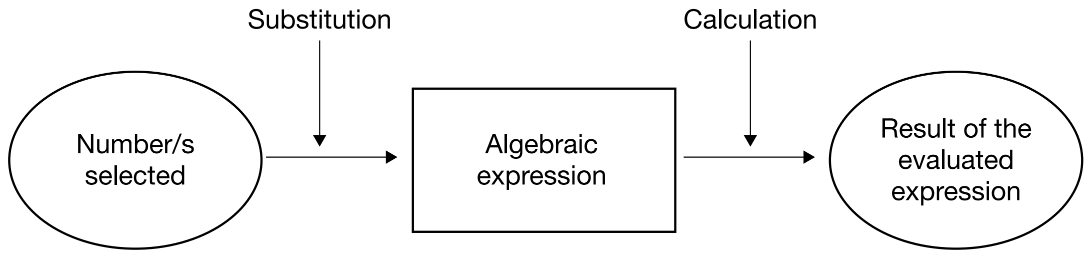


(b)



About the Task

In school algebra, symbolic expressions consist of letters (representing numbers) and operations. Substitution involves replacing a letter by a number, and evaluating the expression for the substituted number (see the illustration below).



The skill of substitution is commonly practiced when studying algebra. This task involves a reverse process: starting from a given result (or a set of results, or a characteristic of a result) and a given expression, and aiming to find the corresponding substituted number or set of numbers. In some cases, the solution can be obtained algorithmically by setting up an equation or informally by trial and error. Some other cases may require applying mathematical considerations, monitoring the solution process, and checking the obtained results.

Comments and Solutions

(a)
- Solving algorithmically the equation $\frac{2}{5}a - 1 = 5$ or applying trial-and-error methods leads to 15 as the required substituted number.
- The expression $\frac{2}{5}a - 1$ yields positive integer results only when substituting multiples of 5. In that case, $\frac{2}{5}a$ will be always even. Thus, any multiple of 5 yields an odd result when substituted in $\frac{2}{5}a - 1$.
- From the previous item, we know that no natural number substituted in $\frac{2}{5}a - 1$ yields an even result.
- The solution of $\frac{2}{5}a - 1 = 0$ ($a = 2.5$) is not a natural number. Substituting 2 and 3 (the closest integers to 2.5) yields $-1/5$ and $1/5$ respectively, as the two closest (and equally distanced) results to zero. Students acquainted with graphical representation of linear functions may sketch the graph of $f(x) = \frac{2}{5}x - 1$, and find graphically the closest integer numbers around the x-intercept.

- By trial and error or by solving the equations $\frac{2}{5}a - 1 = 10$ and $\frac{2}{5}a - 1 = 11$ we find that $a = 27.5$ and $a = 30$ respectively. Thus, $27.5 < a < 30$, and 28 and 29 are therefore the only possibilities.

(b) • The expression $(a - 1) \cdot b$ has two variables, and the substitution process involves ordered pairs of numbers.

- The value of $(a - 1) \cdot b$ will be zero when one (or both) of its factors are zero—that is, when (i) the first number in the substituted pair is 1 (and the second is any of the five given numbers), or (ii) the second number is zero (and the first number is any of the five given numbers).
- An alternative solution method would be to make a comprehensive list of ordered pairs in a 5×5 table (see table below) and to choose from it the pairs that solve the task.

a \ b	–2	–1	0	1	2
–2					
–1					
0					
1					
2					

- When substituting the five given numbers in the expression $(a - 1) \cdot b$, the result of -1 can be obtained only when one of the factors is -1 and the other is 1. Thus, there are only two possible solutions: $a - 1 = 1$ and $b = -1$, namely, $(2, -1)$; or $a - 1 = -1$ and $b = 1$, namely $(0, 1)$.
- In order to obtain a positive odd result, the two factors of the expression $(a - 1) \cdot b$ should be both positive and odd, or both negative and odd. The only possibilities are as follows:

  $a - 1 > 0$ and $b > 0 \Rightarrow a = 2$, and $b = 1$, in which case the value of the expression is 1, or

  $a - 1 < 0$ and $b < 0 \Rightarrow a = 0$, and $b = -1$, or $a = -2$ and $b = -1$.

  In sum, the ordered pairs that solve the problem are $(2, 1)$, $(0, -1)$ and $(-2, -1)$, which yield a result of 1 for the first two pairs and 3 for the third pair.

- A not very efficient method would be to substitute all twenty-five pairs and calculate the corresponding results. Such a procedure allows one to identify both the largest and the smallest possible values. Another way would be to note that the largest possible absolute value for the first factor is 3 (when $a = -2$) and for the second factor is 2 (when $b = 2$ or $b = -2$). Thus, the largest possible value is 6 (when $a = -2$ and $b = -2$), and the smallest is -6 (when $a = -2$ and $b = 2$).

Competencies

Understanding and applying concepts

Both parts of the task require students to apply previously learned concepts and to analyze the structure of the expressions involved. The second part also requires understanding the process of substitution in a two-variable expression.

Reversed thinking

This task requires proceeding backward—from the result (or results) of a substitution to the substituted value(s). Sometimes this reversed thinking may allow for applying an algorithmic solution of an equation or for using non-algorithmic trial-and-error methods. In many other cases, in order to obtain a solution, students must employ further non-algorithmic arithmetic and algebraic considerations.

Generating examples

In the process of solving the task, students may need to produce and examine a collection or an exhaustive list of examples.

Monitoring one's own or others' work

Students can check their results by direct substitution of the obtained results.

Generalizing

For some of the items in this task, students are expected to experiment with several numbers and then define a complete solution set.

Justifying and proving

This task provides several opportunities to produce arguments in order to convince (oneself or others) that all relevant solutions were obtained.

Task 1.3

Creating Algebraic Expressions

Part 1

1. Consider the expression $2 - x$.

Substitute 0. What do you get?

Substitute 2. What do you get?

2. Create an algebraic expression, such that when substituting 0, the outcome is -1, and when substituting -1, the outcome is 0.

3. Choose any three numbers. For each of these numbers, create an algebraic expression such that when substituting the number, it yields 0, and when substituting 0, it yields that number.

Part 2

4. Choose three negative numbers and substitute them in the expression $x^2 - 4x$. Are your results positive or negative? Would your answer be the same for *any* negative number? Explain.

5. Create two other algebraic expressions such that substituting *any* negative number in these expressions yields a positive result.

Part 3

6. Substitute $1/2$, -5, and -0.9 in the expression $x^2 + 2$. Are the results of the substitution smaller or larger than the substituted number? Will your answer be the same for *any* other number? Explain.

7. Create two other algebraic expressions, such that *any* number substituted in these expressions yields a result that is larger than the substituted number.

Part 4

8. Find an expression that represents the perimeter of a rectangle having side lengths of x and 1.5.

9. Give four examples of pairs of expressions for the side lengths of rectangles that have the same perimeter as the rectangle in item 8. Indicate the domain of these expressions.

About the Task

The substitution of numbers into algebraic expressions is a commonly practiced skill in algebra. The tasks in this chapter attempt to integrate both algorithmic aspects of this skill and non-algorithmic reasoning. For example, in task 1.2 (Working Backward: From Results to Substituted Numbers) students are given numerical results or a characteristic of a set of results, and they are required to find the substituted number or sets of numbers. Some of the items in that task allow one to apply an algorithm (solving an equation); yet others require some considerations based on the properties of numbers and operations. In most of the items of this task, both the substituted numbers and the results of the substitution are provided and students are required to create suitable expressions. This is a non-algorithmic task, and students should employ solution strategies related to arithmetical calculations, algebraic and graphical representations, and reasoning based on the properties of operations.

Part 4 of this task is posed in a geometrical context: Students are given an expression that represents twice the sum of two addends (i.e., the rectangle's perimeter) and are required to apply the reversed direction of adding algebraic terms—finding corresponding pairs of expressions that represent the side lengths of a rectangle.

Throughout the different parts of the task, students can suggest (and later discuss) various possible solutions, according to their level of knowledge and creativity.

Comments and Solutions

Part 1

1. This is a simple introductory exercise to set the scene and provide students with possible ideas for the next items. Substituting 0 yields 2, and substituting 2 yields 0.
2. This question can be solved by trial and error or by representing the pairs of numbers (the substituted number and the corresponding result) as points in the Cartesian plane—i.e., considering the points $(0, -1)$ and $(-1, 0)$, searching for graphs that contain these points, and finding an algebraic representation of such graphs (for example, a straight line represented by $-x - 1$, or more complex solutions, such as $x^2 - 1$, $-(x + 1)^2$, or $x^3 + 2x^2 - 1$).
3. The question can be answered at different levels of generality and mathematical sophistication, including the following:

   - Linear algebraic expressions, such as $5 - x$ if the chosen number is 5, or $-7 - x$ if the chosen number is -7
   - Generalized linear algebraic expressions: $a - x$ for any chosen number a
   - Nonlinear expressions. For example, if the chosen number is 2, some of the possible expressions are $x^2 - 3x + 2$, $(2 - x)(1 - x)^2$, or $\sqrt{4 - x^2}$.

   Note that many other expressions can be obtained by employing a functional approach and searching for functions (in our examples, a quadratic polynomial, a third degree polynomial and a semicircle) that take the value of $y = 2$ for $x = 0$ and the value of $y = 0$ for $x = 2$. In a graphical interpretation: there are many graphs of functions that go through the points $(0, 2)$ and $(2, 0)$.

Part 2

4. This property can be explained in various ways:

   - By reasoning based on the properties of arithmetical operations: Substituting a negative number in the first term of the expression (x^2) yields a positive number, substituting it in the second term ($4x$) yields a negative number, and subtracting a negative number from a positive number ($x^2 - 4x$) always results in a positive number.
   - By considering the quadratic function $f(x) = x^2 - 4x$, students can conclude that its graph is in the second quadrant for any negative value of x—that is, the function will have a positive value for all these values of x.

5. As with the answer to item 4 above, students can employ numerical or graphical considerations to find examples such as $-x$, x^2, or $|x|$.

Part 3

6. One needs to consider three separate cases: (*a*) similar to part 2, substituting any negative values of x into the expression yields a positive result that will of course be larger than the substituted number; (*b*) squaring any values larger than 1 for x yields a larger result, and therefore, $x^2 + 2$ will be larger than the substituted number as well; (*c*) squaring any x between 0 and 1 ($0 \leq x \leq 1$) will yield a smaller result, but the obtained result will still remain in the same range. Therefore, the result of the substitution of such a number in the expression $x^2 + 2$ will be in a range between 2 and 3—that is, it will be larger than the substituted number.
7. The problem can be stated as follows: Find an expression of x that is greater than x for any value of x. Thus, for example, the addition of a positive number to x always increases its value (e.g., $x + c$, where $c > 0$). Other possible solutions are $x^2 + 1$ or $|x| + 2$. If students are acquainted with functions, they can restate the problem as follows: Find the algebraic expression for a function whose graph is above the graph of $f(x) = x$ in all its domain.

Part 4

In this section, students create pairs of expressions that produce a given sum.

8. The perimeter of the given rectangle is $2x + 3$.
9. There are several ways to find two side lengths of a rectangle with a perimeter of $2x + 3$ or to search for two expression whose sum is $x + 1.5$. Some possible solutions are as follows: x and 1.5; $x + 1$ and 0.5; $x - 2$ and 3.5, and four side lengths of $\frac{x}{2}+0.75$ (in which case the rectangle is a square). Students should be encouraged to look for examples that have both side lengths represented by algebraic expressions—for example, $2x + 1$ and $0.5 - x$ (and verify that the perimeter is indeed $2x + 3$).

Compentencies

Understanding and applying concepts

Task 1.3 involves numerical and algebraic processes related to substituting numbers in algebraic expressions. Students are expected to understand, apply, and explain the connections between the properties of numbers and those of algebraic expressions. In addition, the task presents an informal encounter with the concept of function for students who have not yet studied it, and an opportunity to apply this concept for those students who had already acquired it.

Divergent thinking

Various algebraic expressions that belong to several different types can all satisfy a given constraint. Thus, the multiple solutions of an item can be categorized (possibly in different ways), defined, and discussed.

Reverse thinking

Creating an expression according to certain substituted numbers and their corresponding results requires thought processes that can be considered as reversed to substituting values in a given algebraic expression.

Generating examples

Throughout the task, students are required to find examples of expressions according to the results obtained by substituting the given numbers. Sometimes the expressions can be grouped into some general categories.

Monitoring one's own or others' work

The solutions of this task can be easily checked by applying the standard substitution process and making adjustments, if needed.

Generalizing

In part 1 of the task, after experimenting with two expressions that have a given property (an algebraic expression, such that when substituting a given number, it yields 0, and when substituting 0, it yields that number), students are required to generalize their findings. In other parts of this task, teachers can encourage similar generalizations—based on various examples presented by their students in a class discussion.