Connections: Looking Back and Ahead in Learning

Most of our discussion has involved the treatment of functions at the high school level. However, functions are also fundamental to mathematics in the middle grades and at the collegiate level. We address these connections in this chapter.

Functions in Middle-Grades Mathematics

Students bring a range of experiences and understandings from the middle grades to their study of functions at the high school level. Middle-grades students develop notions of variable, analyze a variety of patterns of change between variables, and represent and explore relationships by using multiple representations.

Analyzing covariation between variables

In the middle grades, students extend their elementary school experiences, now not only exploring patterns but also generalizing those patterns and developing informal notions of a *variable* as a quantity that changes. At the same time that students are making this transition, middle-grades activities should challenge them to move from the exploration of arithmetic and geometric patterns to the informal study of functions as special relationships between variables. The aim of these activities should be to introduce middle-grades students to some of the understandings related to Big Ideas 1–5.

In the middle grades, most of students' work with functions involves analyzing and representing relationships between two variables from a covariation perspective, as discussed in chapter 1. For example, relationships such as those depicted in the graphs in Reflect 2.1 can offer opportunities for students to describe how a change in one variable relates to a change in another variable (Essential Understanding 2*b*).
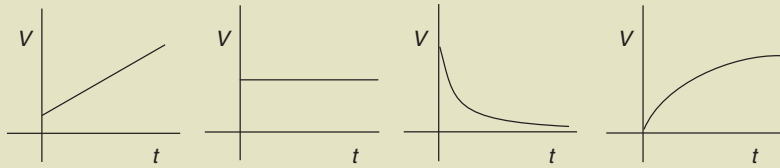
Essential Understanding 2*b*

A rate of change describes how one variable quantity changes with respect to another—in other words, a rate of change describes the covariation between two variables.

Reflect 2.1

Describe how the amount of water in a swimming pool (*V*, for volume) changes over time (*t*) in each case:

Reflect 2.1 asks for descriptions of changes to the *volume* of water in the pool (not the *height* of the water in the pool) over time. Students might describe the first graph as representing a scenario in which a pool is filling with water at a constant rate—as each unit of time passes, the same amount of water is added. The fourth graph also shows that the pool is filling with water; however, in this case, it is filling rapidly at first and then more slowly. The difference between these two increasing graphs allows students to recognize and articulate qualities of both *linear* and *nonlinear* relationships, thus initiating students' formal development of Essential Understanding 3*a*.

The second graph in Reflect 2.1 illustrates a *constant* function, in which the volume does not change over time; the water level in the pool remains constant. The third graph shows a decreasing relationship between *V* and *t* as water is drained from the pool, rapidly at first and then more slowly over time. By creating and examining descriptions of the graphs in this problem, middle-grades students can explore—informally and without extensive use of symbols—how and why changes occur in a particular situation.

Families of functions

Although mathematics in the middle grades explores many families of functions, linear functions receive particular emphasis. Developing students' understanding of the notion that linear functions are characterized by a constant rate of change, as stated in Essential Understanding 3*b*, is a central goal of middle-grades activities. Study of linear functions builds on students' previous work with proportions. Middle-grades students are familiar with many real-world situations in which two quantities are in direct proportion, including examples such as the following:

> A car is traveling at a constant speed; every 5 minutes, the car goes 3 miles.

➡ **Essential Understanding 3*a***

Members of a family of functions share the same type of rate of change. This characteristic rate of change determines the kinds of real-world phenomena that the functions in the family can model.

➡ **Essential Understanding 3*b***

Linear functions are characterized by a constant rate of change. Reasoning about the similarity of "slope triangles" allows deducing that linear functions have a constant rate of change and a formula of the type f(x) = mx + b for constants m and b.

This situation can be described by a linear function, with time t (in minutes) as input and distance d (in miles) as output. The ratio table in figure 2.1 illustrates how d increases by 3 miles as t increases by 5 minutes. The graph of this function is a line though the origin whose points (t, d) satisfy $d/t = 3/5$ (or $d = 3/5 \, t$). The slope of this graph is $3/5$, the constant of proportionality.

Middle-grades students also examine linear functions with nonzero y-intercepts. Consider, for example, a situation in which a plain 12-inch pizza costs $6.99 and each topping costs an additional $0.80. In this situation, the price of a pizza "starts" at $6.99 and increases by $0.80 each time one topping is added. A linear function, represented in three ways (Essential Understanding 5a) as in figure 2.2, helps students to describe and understand this familiar sort of relationship that involves change between variables.

Essential Understanding 5a

Functions can be represented in various ways, including through algebraic means (e.g., equations), graphs, word descriptions, and tables.
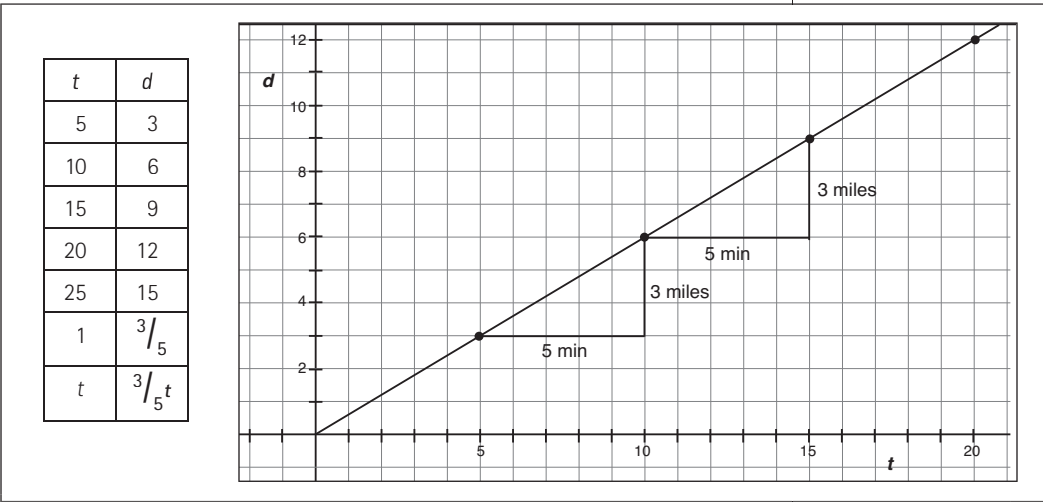

t	d
5	3
10	6
15	9
20	12
25	15
1	$3/5$
t	$3/5 \, t$

Fig. 2.1. A ratio table showing that as t increases by 5 minutes, d increases by 3 miles

$y = \$6.99 + \$0.80x$

Number of toppings	cost of pizza
0	$6.99
1	$7.79
2	$8.59
3	$9.39
4	$10.19
5	$10.99
6	$11.79
7	$12.59

Fig. 2.2. Three representations of the cost of a pizza

Representing relationships with graphs, tables, and rules

In the middle grades, students begin to recognize the usefulness of different representations—particularly graphs, tables, and algebraic rules (Essential Understanding 5c). For example, in the pizza situation, the table allows students to arrive at answers quickly to such questions as, "How many toppings are on a pizza that costs $9.39?" However, the algebraic rule provides the most efficient tool for generating particular input-output pairs (e.g., the cost of a pizza with 4 toppings). The graph offers a clear picture of the linear nature of the relationship between number of toppings and cost. Although the points on the graph in figure 2.2 are not connected, they do lie on a line. Middle-grades students should discuss why the points on such a graph are not connected—it makes sense to talk only about whole numbers of pizza toppings. Students may question the relevance of using large numbers of toppings or raise the possibility of half toppings (that is, toppings that cover only half of the pizza), and doing so reflects a legitimate effort to constrain or extend the domain of this function.

Middle-grades activities also help students to observe how the same characteristics of a particular function can be determined through analysis of different representations (Essential Understanding 5b). In the pizza situation, the *y*-intercept of the graph corresponds to the first entry of the table, when no toppings are added to the pizza. Middle-grades students would also consider how the rate of change is evident in the table, graph, and algebraic formula. These experiences can help students to understand the slope-intercept form of a line, which can sometimes lack meaning for students.

Consider Reflect 2.2, which presents a growing pattern of squares. An important goal in the middle grades is to revisit and generalize patterns explored in the elementary grades, such as the geometric pattern shown.
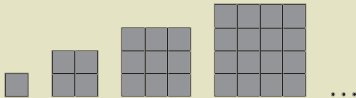
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In the elementary grades, students typically describe this pattern with words and numbers and continue the pattern by drawing or building with colored tiles. In the middle grades, students are equipped to describe changes to the perimeter and area of the squares in the pattern. Their representations of these relationships often include a table, graph, and formulas, as shown in figure 2.3.

As students create these representations, important questions arise: Does it make sense to connect the points on the graph? What is the meaning of the numbers on the *y*-axis when area and perimeter are graphed together (since area and perimeter are measured with different units)? These different representations allow students to

Reflect 2.2

The geometric pattern below contains several different numeric sequences. Each large square is comprised of unit squares.

Consider the pattern in the *perimeters* of the large squares in the sequence. To do so, create a table, graph, and equation that represent the relationship between perimeter and position in the sequence (which is the same as the side length of the square). Use these representations together to describe changes in a square's perimeter relative to its side length.

Create representations of the *area* relationship in the sequence. How does the area of a square relate to its side length? Use multiple representations to explore this question.
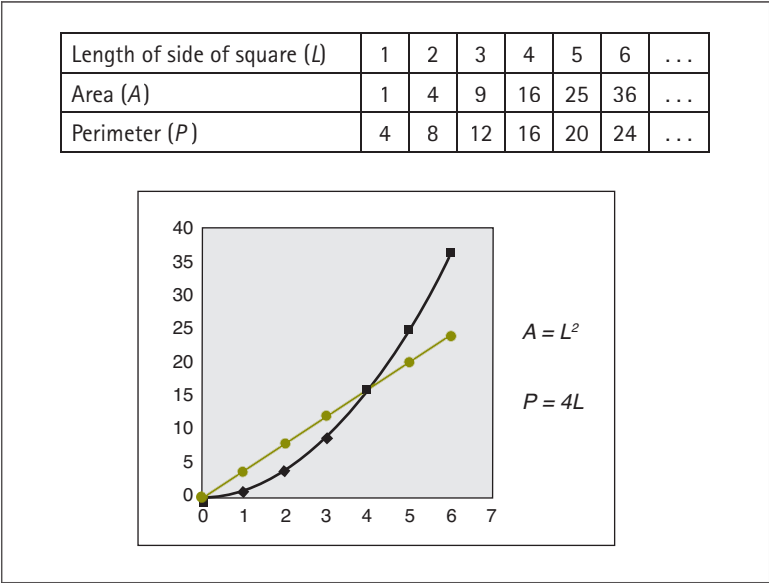
Length of side of square (L)	1	2	3	4	5	6	. . .
Area (A)	1	4	9	16	25	36	. . .
Perimeter (P)	4	8	12	16	20	24	. . .

$A = L^2$

$P = 4L$

Fig. 2.3. A table, graph, and rules to show the relationship between the side length of a square and its perimeter and area

identify specific information about the two relationships—namely, the perimeter and area of a square with a given side length. Students also learn to describe the covariation between side length and perimeter and between side length and area in terms of the different rates of change in the two relationships. Whereas the perimeter function exhibits a constant rate of change, the rate of change in the area function *changes* as the sequence progresses. In addition, students'

ability to use the graph and table to identify the points at which perimeter and area have the same numerical value lays conceptual groundwork for solving algebraic equations.

Much of what students learn about functions in grades 9–12 draws on the ideas and understandings that emerge from problems of the sort discussed in this section. For example, experiences in the middle grades with linear and nonlinear functions in different representations lead to the analysis and classification of many function families in high school. Ultimately, middle-grades experiences should prepare students to use functions as tools to understand and describe change in diverse real-world and mathematical situations.

Connections to Collegiate Studies

One theme that we have examined is that the members of a family of functions share not only the same type of formula, but they also share a characteristic pattern of change. At the college level, the study of "the way functions change" takes place in differential calculus, integral calculus, and differential equations.

Differential equations

A differential equation is an equation that relates a function with some of its derivatives (such as the first or second derivative). For example, Newton's law of cooling states that if $T(t)$ is the temperature of a hot liquid t seconds after it is poured into a container, then the rate at which the temperature decreases is proportional to the difference between the temperature of the liquid and the ambient room temperature. Reflect 2.3 asks you to consider this proportional relationship.

Reflect 2.3

Try to formulate Newton's law of cooling with an equation that involves $T(t)$, its derivative $T'(t)$, the ambient room temperature A, and a constant of proportionality k.

The difference between the temperature of the liquid and the ambient room temperature is $T(t) - A$. A quantity that is proportional to this difference is of the form $k \cdot (T(t) - A)$ for some constant k (which depends on how well or poorly the container insulates the liquid). Because the rate at which the temperature changes is the derivative of the temperature function, the temperature function must satisfy the differential equation $T'(t) = -k \cdot (T(t) - A)$. Why is there a negative sign (assuming k is positive)? The temperature is decreasing, so the rate of change of the temperature function is negative. It turns out that the temperature functions that solve