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The Five Fundamentals

You've probably heard it a thousand times: Kids need to be fluent with basic math facts. You've probably seen the word *fluency* on progress reports, in elementary mathematics standards, and in textbooks, but what does fluency actually mean? For nearly a decade, we've been asking this question to teachers and administrators. Common responses include

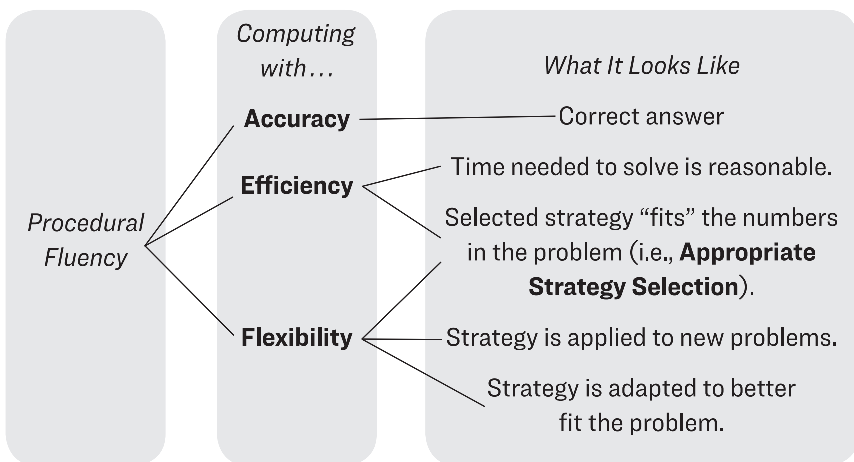
- "They just know the facts."
- "They are fast and accurate."
- "They understand what the fact means."
- "They have strategies to figure out the facts."
- "It's like when you are fluent in a language—you don't have to think or hesitate much."
- "They are automatic with the facts."
- "They can apply their understanding of the facts to new situations."

As you can see, the school community has struggled to embrace a common and comprehensive definition of fluency. Some definitions focus on speed, while others focus on understanding. Reaching the goal of basic fact fluency requires establishing a shared and complete understanding of the term. As baseball great Yogi Berra once noted, "If you don't know where you're going, you'll end up someplace else." This is the tenet behind the first of our five basic fact fundamentals; the four that follow lay out essential elements for designing an effective plan—one that will help every student learn (and remember) the basic facts while building mathematical confidence and number sense.

Fundamental 1: Mastery Must Focus on Fluency

Procedural fluency includes accuracy, efficiency, flexibility, and appropriate strategy selection (National Research Council, 2001). Note that this definition of procedural fluency applies to all operations, not just basic facts, and these elements of fluency are interrelated (Bay-Williams & Stokes Levine, 2017) as illustrated by the diagram in Figure 1.1.

FIGURE 1.1 **What Procedural Fluency Is and What It Looks Like**



The four components (**bolded**) are interrelated. Appropriate strategy selection is required for efficiency and flexibility.

Applying strategies is different than applying algorithms. According to the Council of Chief State School Officers (CCSSO) and National Governors Association (NGA), computation strategies are "purposeful manipulations that may be chosen for specific problems," while algorithms are a "set of predefined steps applicable to a class of problems" (CCSSO & NGA, 2010, p. 85). When students only learn a single procedure, regardless of how quickly and accurately they can implement it, they are denied the opportunity to develop procedural fluency. Strategy selection, adaptation, and transference are critical to both procedural fluency and mathematical proficiency and must be a significant part of students' experiences with the operations right from the beginning, with learning basic facts.

We use these general definitions of each component to focus specifically on basic fact fluency:

- *Accuracy:* the ability to produce mathematically precise answers
- *Efficiency:* the ability to produce answers relatively quickly and easily
- *Appropriate strategy use:* the ability to select and apply a strategy that is appropriate for solving the given problem efficiently
- *Flexibility:* the ability to think about a problem in more than one way and to adapt or adjust thinking if necessary

Consider these aspects of fluency in terms of level of cognition. Which of these requires higher-level thinking? Selecting a strategy, key to both efficiency and flexibility, requires first understanding how and when each strategy is appropriate, and then analyzing a problem to select a viable strategy. Notice that fluency requires understanding, applying, analyzing, and comparing—all higher-level thinking processes. The more students are asked to think at a higher level, the more they learn.

Basic facts are also described in terms of fluency in state and national standards, such as in these examples from the Common Core State Standards (CCSSO & NGA, 2010) (emphasis added):

> **1.OA.6 (Grade 1):** Add and subtract within 20, *demonstrating fluency* for addition and subtraction within 10. *Use strategies* such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$). (p. 15)
>
> **2.OA.2 (Grade 2):** *Fluently* add and subtract within 20 *using mental strategies* [with a reference to 1.0A.C.6]. By end of Grade 2, *know from memory* all sums of two one-digit numbers. (p. 19)
>
> **3.OA.7 (Grade 3):** *Fluently* multiply and divide within 100, *using strategies* such as the relationship between multiplication and division (e.g., knowing

> that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, *know from memory* all products of two one-digit numbers. (p. 23)

These standards acknowledge that it is through the application of strategies that a student develops fluency, and it is through the use of strategies that students come to know their basic facts, or develop *automaticity* (more on this point in the next section). However, the activities and assessments traditionally associated with learning basic facts (such as drill, flash cards, and timed testing) exclusively focus on students' accuracy and one part of efficiency (speed), neglecting strategy development. Many studies over many years have compared traditional basic fact instruction (i.e., drill) to strategy-focused instruction. All of them show that strategy groups outperform their peers on using strategies *and* on automaticity and accuracy (Baroody, Purpura, Eiland, Reid, & Paliwal, 2016; Brendefur, Strother, Thiede, & Appleton, 2015; Locuniak & Jordan, 2008; Purpura, Baroody, Eiland, & Reid, 2016; Thornton, 1978, 1990; Tournaki, 2003). We *know* that strategy development is absolutely necessary for fluency. And fluency is essential to developing automaticity with basic facts.

Fundamental 2: Fluency Develops in Three Phases

As students come to know basic facts in any operation, they progress through three phases (Baroody, 2006):

- **Phase 1: Counting** (counts with objects or mentally)
- **Phase 2: Deriving** (uses reasoning strategies based on known facts)
- **Phase 3: Mastery** (efficiently produces answers)

Consider these phases in the context of mastering addition facts. Most students enter kindergarten or 1st grade using counting to solve addition or subtraction problems. They may be counting with objects, on their fingers, or in their heads, but, regardless, these students are still considered to be at Phase 1. As they start to learn some of the easier facts (usually 2 + 2 = 4, 3 + 3 = 6, and 5 + 5 = 10), they can begin using those facts to help them to figure out more difficult, related facts. For example, to find 5 + 7, a student might begin with 5 + 5 = 10 and add on two more to determine that 5 + 7 = 12. This is an example of Phase 2 thinking, where the answer to a more challenging fact is being derived by using a known fact. The flexibility, increased efficiency, and selection of appropriate strategies that are developed in this phase are critical to fluency.

As students engage in sufficient meaningful practice in Phase 2, they become faster in their strategy selection and application and come to know some facts without needing to apply a strategy. Thus, they move naturally into Phase 3 (mastery), which is characterized by the highly efficient production of answers, either through quick strategy application or through recall. Students operating at Phase 3 are considered *automatic* with those facts, as they meet the definition commonly accepted for automaticity—answering within three seconds, either through recall or automatic strategy application (Van de Walle, Karp, & Bay-Williams, 2019). Thus, the difference between Phase 2 and Phase 3 is essentially speed; in both cases, students may be applying appropriate strategies flexibly, but students at Phase 3 answer instinctively within a few seconds, whereas Phase 2 students might take longer to select and apply a strategy.

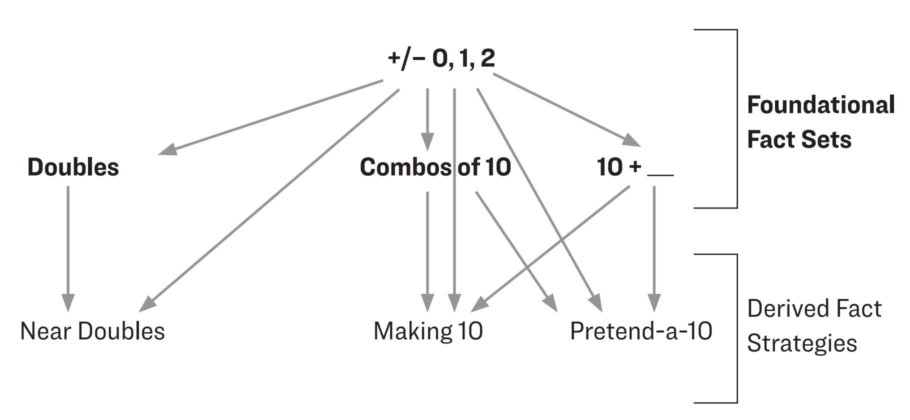
You've likely seen advertisements for books or fact-learning programs that promise "fact fluency in two minutes a day" or "know your facts in seven days." The reality is there are no shortcuts to developing fluency or to mastery and automaticity. "Quick fix" programs attempt to take students who are operating at Phase 1 (counting) and push them directly to Phase 3, usually through drill and timed testing, skipping any effort to explicitly teach strategies or focus on number relationships. Students subjected to such programs may appear to know the facts in the short term, but within weeks or months they are back to where they started: counting. Because little to no time is spent in Phase 2, once facts are forgotten, students have no efficient, appropriate, and flexible strategies to fall back on. This explains why we sometimes see middle grade students counting to solve basic facts, much to the chagrin of their teachers. In contrast, to encourage lasting mastery of basic facts, students need to have sufficient time and experiences in Phase 2. The activities, games, and assessment tools you will find in this book are designed to do just that.

Fundamental 3: Foundational Facts Must Precede Derived Facts

Perhaps you have memories of learning groups of multiplication facts in order. You memorized the 0s facts, passed a test (and perhaps got a sticker on your chart), and then moved on to the 1s, 2s, 3s, and so on. Although once common, this sequence is not consistent with what research suggests is the most effective approach to learning facts. There are sets of facts within both addition and multiplication that are easier for students to master first and are essential to applying derived fact strategies. We refer to these facts as *foundational fact sets*, or *foundational facts* for short. A foundational fact set is a set of facts that illustrate a specific pattern or number relationship. For example, working on the *one less* facts can be connected to the counting sequence (the number that comes before), to the number line (the number that is one to the left), and to the idea of taking away one.

The remaining facts can be derived from the foundational facts through strategy application. Thus, these sets of facts are called *derived facts*, and students come to know these facts by learning *derived fact strategies*. Notice that we do not use the term *subset* with derived facts. That is because many derived facts can be reasonably solved using more than one derived fact strategy. In fact, students must have many opportunities to select which of the derived fact strategies they will use to solve a combination that they do not know. A flexible learning progression demonstrating the relationships between facts for addition is presented in Figure 1.2. Many studies have found that mathematics teaching based on learning progressions leads to positive effects on children's early math achievement (Frye et al., 2013).

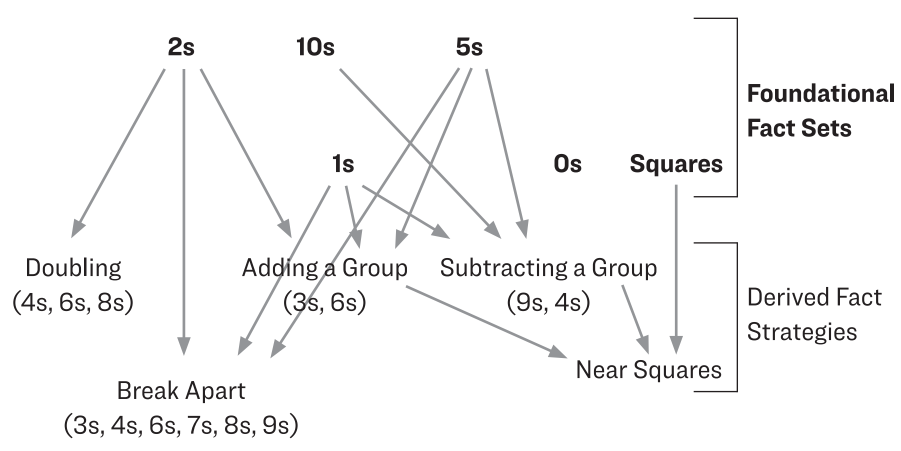
FIGURE 1.2 **Addition Fact Fluency Flexible Learning Progression**



In this chart we include the +/– 0, 1, and 2 facts as foundational and the place to begin. Notice that each of the other foundational facts, except perhaps doubles, can flow easily from already knowing +/– 0, 1, and 2. Therefore, to work toward mastery of all facts, a first step is to develop automaticity with the +/– 0, 1, and 2 facts. At the next level are more foundational facts, which can be taught in a flexible order, as mastery of one is not needed to reach mastery of another. However, students must master specified foundational facts to use the related derived fact strategies on the final level (e.g., Doubles must precede Near Doubles).

Similarly, multiplication facts can be taught in groupings so that known foundational facts can be used to derive other facts. The flexible learning progression for multiplication is shown in Figure 1.3.

FIGURE 1.3 **Multiplication Fact Fluency Flexible Learning Progression**



The distinction between foundational fact sets and derived fact strategies is essential for effective teaching of the basic facts because it provides a blueprint for monitoring fact instruction progress. Consider the example of finding 5 + 7. If a student didn't already know that 5 + 5 = 10, he would not be able to arrive at the solution as described: 5 + 5 = 10, 10 + 2 = 12, so 5 + 7 = 12. Thus, when we observe students who are unable to use a strategy for finding this fact, we must determine if they have learned the foundational facts to automaticity. If not, that is where intervention must be focused.

Because this progression is so important to students' success with using strategies to master basic facts, we have organized this book around the two groupings of foundational facts and derived fact strategies. Chapters 2 and 4 focus on teaching the foundational fact sets for addition and multiplication, respectively, and Chapters 3 and 5 do the same for derived fact strategies. Chapters 6 and 7 focus on assessing foundational fact sets and derived fact strategies, respectively. The two flexible learning progressions will appear throughout the book to highlight the fact sets or strategies discussed. Our hope is that these charts will not only help you visualize the progression of fact mastery in a typical classroom but also help you with individual progress monitoring in order to develop plans of action to support students who have not yet mastered all the basic facts.

Fundamental 4: Timed Tests Do Not Assess Fluency

Picture a worksheet containing 100 multiplication facts in random order, which students are asked to complete in five minutes. Perhaps you remember these timed tests from your childhood, or perhaps you still see these in use in classrooms today. Now determine how many of the four components of fluency (flexibility, accuracy, efficiency, and appropriate strategy use) you believe are actually assessed with a timed test. We've posed this task countless times to many groups of teachers and administrators, most of whom have initially thought that, at most, two components are assessed—but which two? Flexibility and appropriate strategy use are easily eliminated. Because the teacher only sees a recorded answer, it is impossible to assess if a student is flexible or chooses appropriate strategies from a timed test alone. This does not mean that students aren't flexible or that they don't use appropriate strategies; *the timed test simply doesn't allow a teacher to see it*. What about efficiency and accuracy? Although at first glance it may seem that a timed test can assess these components, there are certainly instances where this isn't true. Consider the following scenarios.

> Tommy is taking his weekly multiplication test. Although he has learned many easier multiplication facts, Tommy still struggles to remember his 7s, and he is very aware of this weakness. Once again, he compensates by skipping around and answering the facts he knows; then he quietly puts his hands under his desk to help him count to answer the remaining, unknown facts. He has learned that he can usually finish the test in time by doing this, and his teacher is therefore convinced he knows his facts.

> Ellie feels her heart start to race when her teacher announces it is time to start the weekly addition facts timed test. Even though Ellie excels at reading, writing, and solving even the most challenging story problems, as soon as the timer starts, she draws a blank. She struggles to remember the facts she knows well and is so distracted by the timer that she can't apply her favorite strategies to tougher addition facts, like 7 + 8. With tears in her eyes, she once again turns in an incomplete test and tells her friends she's "just so bad at math." Her teacher is puzzled; she has seen Ellie's automaticity with addition facts many times during math games and doesn't understand why that doesn't translate to the test.

Whether it be from our own childhood experiences or from experiences as an adult, we've all known students like Tommy and Ellie. Let's look at Tommy's fluency. Even though he knew some of the facts, the completed, correct answers on his timed test give the illusion of mastery. The reality is that he is not efficient with all the facts (namely, the 7s), and yet his ability to "play the game" has not only fooled his teacher but also reinforced that he doesn't need to make an effort to learn those facts. Tommy's case illustrates how timed testing does not provide a wide enough lens for evaluating fluency, because it doesn't reveal the exact facts with which students are efficient.

Next, consider Ellie's fluency. She is an excellent mathematical thinker, loves to write and solve problems, and, based on her teacher's observations during game play, has mastered the addition facts. Yet the pressure of time cripples her thinking and essentially invalidates her test as a measure of accuracy. Even worse, this experience has convinced Ellie that she is bad at math when, in fact, she is quite the opposite! Ellie is not unique. In fact, over the past decade there have been numerous findings from psychology and even neuroscience uncovering the damaging effects of timed testing. For example, in a study of more than 50 students in 1st and 2nd grades, Ramirez, Gunderson, Levine, and Beilock (2013) found that students begin experiencing math anxiety as early as 1st grade and that anxiety was not correlated with reading achievement or socioeconomic status. However, they did find an important, troubling correlation: The students who tended to use more sophisticated mathematical strategies were those who often experienced the most negative impact on achievement due to math anxiety. In other words, by age 7, many young students with high mathematical aptitudes are already learning to fear math. Boaler (2012, 2014) also reported that even students who perform well on timed tests share concerns such as "I feel nervous" and "I know my facts, but this just scares me."

Timed testing is often considered synonymous with learning basic facts, and yet, as we have just described, it is highly ineffective at assessing any of the four components of fluency. Why, then, is it still so common? Some schools feel that timed testing is necessary for promoting fact mastery. However, there is no evidence to support this theory. In fact, there is evidence to the contrary. In a study of nearly 300 1st graders, Henry and Brown (2008) found that those students who were more frequently exposed to timed testing actually demonstrated slower progress toward automaticity with their facts than their counterparts who were not tested.

Why are timed tests still so prevalent given the evidence that they don't work? We think many schools continue to use timed testing because they simply do not know how else to assess fact mastery. We hope to rectify this issue and offer a variety of formative assessments in Chapters 6 and 7. These assessment tools and techniques allow teachers to assess all four components of fluency while still encouraging mathematics confidence in their students.

Fundamental 5: Students Need Substantial and Enjoyable Practice

Substantial and enjoyable practice should be considered as an alternative to timed drills for developing mastery with basic facts. Imagine, for example, posing this question to 1st or 2nd grade students: *How many equations can you write that equal 10?* Students enjoy open-ended challenges, and the task allows for natural differentiation, with some students inventing equations with three and four addends and others simply listing known facts. In looking at the ways the students thought about their equations, you learn what number relationships they know—and, as you will soon read, knowing how far numbers are from 10 is an essential concept.

These same students may also play Go Fish for 10s, a game where a match is a combination of 10 (a student with a 4 asks, "Do you have any 6s?"). As you will see later in this book, many familiar games can be adapted to practice basic facts, including Four in a Row, Concentration, and War. We will also share many novel basic fact games, such as Crossed Wires, in which students create grids (arrays) of crossed wires to practice derived fact strategies for multiplication.

Games and other enjoyable challenges provide ample fact practice without constantly using those pages with 100+ facts. Additionally, games are *interactive*, so students can think aloud and hear others' strategies (Bay-Williams & Kling, 2014; Godfrey & Stone, 2013). Think-aloud opportunities are beneficial to all students but are particularly effective with students who traditionally struggle to learn mathematics (Frye et al., 2013; Gersten & Clarke, 2007).

Furthermore, during game play, the teacher has an opportunity to implement formative assessment tools to monitor each student's progress toward mastery. However, it is not enough for a game to be fun; it also has to provide a meaningful mathematics experience. The features described below provide guidance on how to select games that will, in fact, provide effective fact practice. Although a game may not reflect all these features, any game that has most of these features will more effectively help students' fluency development.

10 Questions to Guide Game Selection

To what extent does the game . . .

1. Provide an opportunity to practice the subset of facts that the students are learning?
2. Appeal to the age of your students?
3. Employ visuals or tools (such as ten frames, quick looks, or arrays) to support strategy development?
4. Involve selecting from among derived fact strategies (for mastery-level games)?
5. Provide opportunities for discussion among students about their mathematical thinking?
6. Encourage individual accountability? (For example, are students solving their own facts or competing to solve the same fact? The former practice provides more "think time" and avoids opting out.)
7. Remove time pressures?
8. Involve logic or strategic moves, enhancing the "fun factor"?
9. Offer opportunities for adaptation so that all students can experience appropriate challenge?
10. Lend itself to you being able to listen and watch in order to assess progress?

Sometimes you need a game that is focused on one set of foundational facts (e.g., 5s facts for multiplication); at other times, you need a game that requires that each student identify a derived fact strategy (e.g., deciding how to break apart one factor to use known facts). When working on derived facts, students need think time. Therefore, when selecting a game to help students practice derived fact strategies, a good choice is one in which time is not a factor and where each player is finding a different fact so that students are not trying to find an answer faster than their partner. Once students are automatic with all of their facts, foundational and derived, then games that involve speed may be appropriate and enjoyable. As students get older, the more strategic a game is, the more fun it is to play. Games like Connect Four, for example, involve trying to both get four in a row yourself and block your partner. In summary, good game selection requires various considerations: age of the student, the facts being learned, and student fluency with that set of facts.

Within the covers of this book are more than 40 games that reflect, at least to some extent, the features of effective games described above. Nearly all of these games are readily adaptable to other fact sets or operations, resulting in well over 100 versions that provide enjoyable, targeted, strategy-focused ways to move students toward Phase 3 (mastery). Additionally, many other games exist online and in various books. The features above can be used to evaluate the quality of these games in supporting students' emerging fluency and automaticity.

Let's Get Started!

Hopefully this chapter has piqued your interest for the need to change how fact fluency is developed and assessed. We have a lot of work ahead! By the time you have finished reading this book, you will

- Develop an understanding of foundational facts for each operation.
- Develop an understanding of research-based derived fact strategies for each operation.
- Learn how to sequence facts instruction to best promote natural strategy development and eventual fluency and automaticity.
- Explore activities and games for helping students progress through the phases of fact mastery.
- Consider a variety of assessment tools that can monitor fact mastery in informative and supportive ways.

Let's get started!