Introduction

How quaint the ways of Paradox!
At common sense she gaily mocks!
W. S. Gilbert

This book is a collection of paradoxes, fallacies, and brain teasers suitable for use in the high school mathematics curriculum. Each of the activities in this book will encourage students to look at familiar mathematical concepts in a new light and deepen their understanding of those concepts.

A paradox is the mathematician's finest joke. A good paradox develops easily, plausibly, and apparently logically until it reaches a punch line that is entirely unbelievable, gaily mocking at common sense. A good paradox, like a good joke, needs to be well told and will provoke anything from an explosive guffaw to a wry smile, followed by a furrowed brow.

As Martin Gardner has observed, paradoxes are also like conjuring tricks. The conjurer, after showing the audience an empty top hat, waves a magic wand and pulls a rabbit out of the hat. How was the trick performed? Where did the rabbit come from? How is it possible that a simple chain of reasoning can lead to the conclusion that one equals zero?

But mathematics is not like magic: conjurers never explain their tricks, yet mathematicians try to explain everything down to the last detail.

Some paradoxes are easily explained because they rely on a simple misuse of a fundamental rule. Perhaps these simple fallacies are not worthy of being called paradoxes at all, merely false arguments. Nevertheless, discussion of simple fallacies goes to the very heart of mathematics: one false assumption may render an entire argument fallacious. The story is told that English mathematician Bertrand Russell (1872–1970) asserted at dinner that a false statement can be used to prove anything. His dinner-table companion challenged him:

"Well, Russell, if I allow you the premise that one is equal to two, can you prove that I am the pope?"

Russell thought for a moment and replied:

"You and the pope are two people. But one is equal to two, so you and the pope are one person. Therefore, you are the pope."

Other paradoxes rely on the fact that the popular use of terms such as *average* or *probability* is sometimes at variance with their strict mathematical definitions. Here we are at the central problem of applied mathematics: fitting our mathematical structures to real-life situations.

Still other paradoxes are very deep and illustrate that the essence of a really good paradox is that no one explanation is ever entirely satisfactory. For example, paradoxes of infinity are especially difficult to unravel, perhaps because the concept of infinity is completely abstract and outside our real-world experience. The great paradoxes of antiquity, such as the paradox of Achilles and the tortoise, were paradoxes of infinity.

Paradoxes are fun but must also be taken seriously, for through the ages paradoxes have been a strong force driving mathematicians to think seriously about the foundations of their subject. The common-sense philosophy of the Pythagoreans that all of mathematics could be encompassed within a theory of whole numbers was mocked by the discovery that the square root of two is not a whole number and cannot be expressed as a quotient of two whole numbers. In relatively recent times the development of calculus was influenced by Irish philosopher Bishop George Berkeley (1685–1753) mercilessly attacking English mathematician Isaac Newton (1642–1727) and German mathematician Gottfried Leibniz (1646–1716) with his paradoxes of infinitesimals. And later, the attempt made by German mathematician Gottlob Frege (1848–1925) to base mathematics on set theory was destroyed by Russell's paradox of the set of all sets that are not members of themselves.

Mathematics is full of paradoxes and can never be free of them. That, paradoxically enough, can be rigorously proved. In the 1930s Austrian logician Kurt Gödel (1906–1978) proved that in any mathematical system that includes ordinary arithmetic there are statements that are undecidable: they make perfect sense but can be neither proved nor disproved. Gödel's theorem proved that the mathematician's work can never be finished. That is the ultimate paradox.

Using this book

A paradox used as a teaching device exploits the idea of cognitive dissonance. When you are presenting the paradoxes in this book to students, we suggest that you remain silent and refrain from intervening in order to allow students the space to struggle with the cognitive dissonance on their own. Don't deprive students of the pleasure of resolving a conflict themselves. A risk in using cognitive dissonance as a teaching technique is that some students may feel confused, frustrated, or deceived. Try to be sensitive to the frustrations of those students who are losing hope and need a hint.

All the one- and two-page activities in this book are student directed and can be used in algebra, geometry, trigonometry, statistics, or calculus classes. The student activities can be downloaded at www.nctm.org/more4u. The activities can be used to reinforce, refine, or clarify a concept your students are studying; to add an element of surprise to your mathematics class to stimulate problem-solving skills; and as discussion topics and extra-credit assignments. Most of the activities can be completed in less than one class period and require no additional materials, although several of the book's geometry activities can be enhanced by the use of geometry computer programs such as The Geometer's Sketchpad® or GeoGebra (www.geogebra.org).

Each activity is accompanied by detailed teacher's notes. The explanations in the teacher's notes are limited to the activity and are meant to be shared with students. The comments for each activity are meant for you, the teacher, and are mainly pedagogical and mathematical. For many activities, related historical material follows the comments. Extensions to the activities are often included in this historical material as well as in the comments. Also included in each set of teacher's notes is a list of the activity's key concepts. A matrix of these concepts at the beginning of the book allows you to easily place each activity in a suitable context. Additionally, a list of recommended reading at the beginning of the book guides you and your students to a wealth of related material.

Acknowledgments

The activities in this book were collected over many years of teaching math and have been used in popular lectures for high school students and in more formal seminars for postgraduate students. Thanks are due to many students and colleagues, math teachers, math educators, and mathematicians for many of the ideas for activities we have included in this book.

Although most of the work on this book was completed via the magic of electronic communication, we would like to express our gratitude to the Technion-Israel Institute of Technology, the Israel Ministry of Science, the University of Cape Town, and the South African Foundation for Research Development for supporting our visits to each other so that we could write earlier drafts while on the same continent.

The first publication of this book benefited from the help of Steve Rasmussen, president of Key Cirriculum Press, and the energetic and professional work of our editor, Greer Lleaud, and her staff. In preparing this updated publication, we are grateful for the helpful comments of the NCTM's publications manager, Myrna Jacobs, and reviewer Albert Goetz.

Nitsa Movshovitz-Hadar and John Webb