CHAPTER

7

Classrooms Where Children Learn

Catherine Twomey Fosnot and Timothy J. Hudson

A teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

—George Polya (2004, p. v)

Don't just read it; fight it! Ask your own questions, look for your own examples, and discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

—Paul. R. Halmos (1985)

Anthony stares at his list of orange box dimensions: $(1 \times 24) \times 1$, $(2 \times 12) \times 1$, $(4 \times 6) \times 1$, $(8 \times 3) \times 1$, $(2 \times 6) \times 2$, $(4 \times 3) \times 2$. He wonders if he has made all the arrangements that would hold twenty-four oranges. Could there be more, or were there really only six unique boxes?

Anthony and Jeremy, his math partner, have been working for the last hour snapping together multilink blocks into rectangular prisms, each out of twenty-four cubes. At first they had started randomly removing and rearranging cubes until they found an overall shape that worked with exactly twenty-four cubes. At the suggestion of their teacher, Kari, they had placed parentheses around the factors showing the bottom layer so their classmates could tell how each box was to be held. Eventually they had hit upon a halving-and-doubling strategy, which worked nicely with the 24. Breaking a shape that worked in half and then rearranging the halves produced a new box—for example, turning a $(24 \times 1) \times 1$ into $(12 \times 2) \times 1$. Now they had six boxes, and it seemed that no matter what they did, they could not find a different one.

"I think we must have them all," Jeremy offers tentatively. "We halved and doubled, and I can't think of any more."

"But we need to be sure," Anthony declares with conviction. "I know if we flip or turn these it seems like we get a different one, but really the box is the same. It's just that the top is different and the numbers are in different places. They will still cost the same to make; they need the same amount of cardboard." Although Anthony is not explicitly aware that he is describing cases of the commutative and associative laws, those properties do underlie his observation.

"What if we try to do it by the number of layers?" Jeremy offers a systematic approach—a powerful insight that will eventually let them develop a proof by all cases.

Mathematicians in Residence: A Summer Intervention and Professional Development Project

Anthony and Jeremy have just completed the fifth grade and are in a summer intervention project called Mathematicians in Residence (MIR; Denise Pupillo and Tim Hudson [Parkway School District], Angelene Hayes and Kathleen Taggart [St. Louis Public Schools], and Sam Hausfather [Maryville University] designed the project. Pupillo and Hudson are the co–principal investigators of this grant-funded program). Funded through the State of Missouri by a federal Mathematics–Science Partnership grant, it is a collaborative effort of the Parkway School District in suburban St. Louis, Missouri; the Saint Louis Public Schools; local parochial schools; and Maryville University. Every summer, approximately 150 middle school students (selected because they scored slightly above or slightly below "proficient" on their most recent state mathematics achievement test) are invited to attend a two-week summer program taught by middle school math teachers from the partnership schools, two math educators (Cathy Fosnot and Kara Imm), and a professional mathematician (Bill Jacob from the University of California, Santa Barbara). In contrast to a deficit model of intervention, which diagnoses needed skills and gaps in understanding and follows with remediation, MIR's philosophy is to treat children as developing mathematicians by inviting them to engage in mathematical inquiry. Emphasis is placed on using realistic situations for investigations and carefully crafted strings of related problems in minilessons to develop computational fluency and on publishing and proving one's thinking to peers in gallery walks and math congresses. The participating middle school teachers commit to attending a one-week professional development program that Fosnot, Imm, and Jacob lead, held each summer the week before the children come. The curriculum materials include *Contexts for Learning Mathematics* (Fosnot et al., Heinemann Press) and *Mathematics in Context* (Encyclopedia Britannica, developed by the University of Wisconsin and the Freudenthal Institute).

The box problem

Anthony and Jeremy are immersed in a series of investigations from *The Box Factory*, a fifth-grade unit on volume and surface area from *Contexts for Learning Mathematics.* Cathy, Kara, and Bill are working alongside the teachers as an integral part of the professional development. The children are exploring the amount of cardboard needed (the surface area) for each of the different boxes that hold twenty-four oranges. Finding the unique boxes challenges some students, whereas others find them more easily. To find the surface area, some cut six rectangles from 3/4-inch graph paper (this size matches the face size of the multilink cubes they used to build the boxes), one for each face, and then count the total number of squares. Other students just count the faces of the cubes showing on each side of each box, either by skip counting or multiplying, and then add the six partial products. Some do not realize that opposite sides of each box have the same area; others do. Differentiation happens naturally in this classroom as teachers sit and confer with students—listening, noting their strategies, and challenging them appropriately as they work.

Back to the classroom

"Yeah, that's a good idea," Anthony responds appreciatively to Jeremy's idea of working systematically, one layer at a time. Starting with one-layer boxes, he produces four unique boxes: $(1 \times 24) \times 1$, $(2 \times 12) \times 1$, $(4 \times 6) \times 1$, and $(8 \times 3) \times 1$. "That's it for one layer, because everything else is a flip or turn, and we have to make twenty-four," he states with conviction. "Now, let's do two-layer boxes."

"We have to make a layer of twelve because 2×12 is 24," Jeremy offers. He produces two more boxes: $(2 \times 6) \times 2$ and $(4 \times 3) \times 2$. "There's only one more way to make twelve in a layer, a $(1 \times 12) \times 2$. But that is just the $(2 \times 12) \times 1$ box flipped around, so I think we have them all. Now you do the three-layer ones. This is a good strategy. I think we are going to find more boxes."

The importance of insight

The mathematician Paul Dirac once commented, "It seems that if one is working from the point of view of getting beauty in one's equations, and if one has really a sound insight, one is on a sure line of progress." The boys are now on a sure line of progress. They have a clear sense of direction and see beauty in their systematic approach. They continue with enthusiasm. Anthony quickly starts to make three-layer boxes by finding ways to make the bottom layer with eight cubes, $(8 \times 1) \times 3$ and $(2 \times 4) \times 3$, only to find that they are repeats as well; they can also be made by flipping other boxes. For example, $(8 \times 3) \times 1$, a box they already had, can be flipped to make $(8 \times 1) \times 3$.

"I bet the same thing will happen if we do four-layers," Jeremy excitedly conjectures. "They will all be repeats, because there's only two ways to make the bottom layer of six, just six times one and three times two, and we have those. Maybe we were right! Maybe there are only six boxes in all."

"Yeah … because we can't do five-layer boxes, it won't come out right. You can't divide 24 by 5 without a remainder. And you can't do seven layers either." Anthony begins to appreciate that factors of 24 are important, and as the boys continue exploring each possibility, they realize that their original list was indeed correct: only six unique boxes are possible. But now they have more than a solution: they have *proven* it to themselves and have a way to convince their peers as well.

Elated from their success and confident that they have found all possible box designs, they carry on exploring how much cardboard they need for each design to determine which box is the cheapest to make. Jeremy cuts a 2×6 rectangle from 3/4-inch graph paper, explaining as he works, "I'm doing the $(2 \times 6) \times 2$ box." Anthony looks again at his list and at the first of six individual rectangles that Jeremy is cutting out and, tentatively, he offers a new idea: "We could wrap the sides where it is the same height, and then just add the bottom and top."

"What do you mean?" Jeremy asks with a puzzled look.

"We could just cut three rectangles instead of six," Anthony states his idea again as Kari joins the boys to confer with them as they work. He beams at Kari and explains, "Jeremy was cutting out each side of the box and I noticed that instead of cutting each side, we could wrap the paper around the sides to get one big piece and we wouldn't have to cut so much."

"That's really interesting, Anthony. What a good idea!" Knowing his idea for finding the lateral surface area is an important insight—one that will help develop a general formula for surface area of prisms—Kari first celebrates his idea, makes sure that Jeremy understands what Anthony means, and then challenges, "What are the dimensions of your rectangles, and how much cardboard is needed for each?"

Anthony has not yet realized that $l \times w$ will produce the area of the rectangle. Tediously he counts along the edges to determine the length and width of the top face, announcing that the dimensions are 2 and 6, and then he begins to count again—this time every square of the rectangular face to determine the area.

Kari stops him. "Wait, you got 2 and 6 for the dimensions, right? Before you count all of the squares, would those numbers help you know how many squares there will be?"

Anthony looks at her quizzically. "What do you mean?"

The importance of invention and supporting development

Many teachers at this point would probably explain that the formula for the area of a rectangle is $A = l \times w$ and ask what the product of 2 times 6 is. But emphasizing procedures over invention and understanding can cause confusion and gaps in students' long-term ability to transfer their knowledge to new situations. How often have we seen children confuse formulas for perimeter with those for area and use $l \times w$ for nonrectangular parallelograms instead of $b \times h$? Instead of offering the area formula, Kari attempts to get *underneath Anthony's thinking* and supports *him* to develop one. Her approach as a facilitator and coach of learning is grounded in the research published in *How People Learn*: "the teacher must actively inquire into students' thinking, creating classroom tasks and conditions under which student thinking can be revealed. Students' initial conceptions then provide the foundation on which the more formal understanding of the subject matter is built" (Bransford 2000, p. 18). Kari starts by bringing him back to what he knows and building from there. "Show me how you got 6 again."

Obliging, Anthony points to the six squares along the edge of the length.

"Oh, that's interesting. There are six squares in a row? You used the edges of them to figure out how long the length was?" Kari works to make the relationships he has used more explicit, encourages him to examine them, and then pushes for generalization. "What an interesting idea. Will the length always have a row of squares of the same number as the length?"

Anthony ponders the question, and slowly he becomes confident that the idea is generalizable. "Yes, because that's what I counted."

"You did. You used the edge of each to calculate the length." Kari goes on, "How many rows of six are there?"

"Two. There's another row on top," Anthony responds.

"That's interesting," Kari says. "Two was the width, too. Is there a relationship here? Are you saying we have two rows of six, two sixes?"

"Yeah … that's 12! We don't have to count," Anthony exclaims with delight.

Interpreting an array in rows and columns is often not an easy feat for children. When asked to draw a picture of a grid they struggle to do so, not realizing that each square is simultaneously in a row and a column (Battista et al. 1998). Understanding the relationship of dimension to area is even more difficult because it requires an understanding that the length of the edge of any given square unit along the perimeter is actually a linear measurement and that these linear units are being iterated to form the overall dimension of the shape. Without this understanding, formulas for area and perimeter make little sense.

Kari now challenges the boys to examine the dimensions of the bigger rectangle—the lateral surface area: "What are the dimensions of this wraparound piece, I wonder?" She offers the challenge, not as a test question or as a question to lead them to her answer of a formula for lateral surface area. Instead, she offers an inquiry, an invitation to wonder with her. "I wonder if there is any way we could have known what the area would be before we cut it out." Even her use of the plural "we" suggests that she is enjoying doing mathematics with them. That she did not pose the question within a discovery model of instruction—where the intent is to discover the teacher's answer—implicitly communicates that mathematics is about wondering, puzzling, and reflecting on mathematical relationships and generalizing. Subtly she is mentoring them, working with them as developing mathematicians.

Jeremy lays the rectangle flat, pondering the question. Anthony sees a connection to his earlier idea. "I think it's like what I said before, but I see another relationship too. Yeah … that's it! It's the distance around the box! That's the long side. The short side is the number of layers, the height! If we add the length and width twice, that is the length of the piece of paper that goes all around. So then just multiply that by the height," Anthony exclaims. He writes "$l + w \times 2 \times h + t + b$," using letters to represent length, width, height, top, and bottom, respectively. Although his notation does not show the standard order of operations, his thinking is solid.

Celebrating accomplishments

Kari celebrates their accomplishment, "Wow. That's a wonderful noticing! Are you saying we could just measure around the box—use the perimeter,