TEACHER'S NOTES

KEY CONCEPTS

- Dividing by zero
- Equations

1

One Equals Zero

Explanation

The error occurred when the factor $x - 1$ was canceled. Because $x = 1$, $x - 1 = 0$, and it is obviously incorrect to deduce from the statement $0 \cdot (x + 1) = 0$ that $x + 1 = 1$. Canceling factors without considering the possibility that a factor could be zero is a common error.

Comments

- Cancelation is division of both sides of an equation by the same nonzero factor. This activity serves as a warning against dividing both sides of an equation by a factor that could be zero. The purpose of this activity is to accentuate the danger of forgetting that cancelation is defined as dividing both sides by a *nonzero* factor.
- When students have completed this activity, they may consider the formal reason for ruling out division by zero of any nonzero number and for ruling out division by zero of zero, which are very different from each other. The reason in both cases is that division by zero cannot be well defined. Division of a nonzero number by zero is undefined because there is no solution to the equation $a \cdot 0 = b$ if b is *not* zero. If b *is* zero, there are infinitely many solutions to $a \cdot 0 = b$ and division by zero cannot be *uniquely* defined, which is the same as saying that it is not well defined.

See Also

See also Activities 2, 3, 7, 15, and 26.

2 Four Equals Five

Carefully read the following:

$$16 - 36 = 25 - 45 \tag{1}$$

$$\Rightarrow \quad 16 - 36 + \frac{81}{4} = 25 - 45 + \frac{81}{4} \tag{2}$$

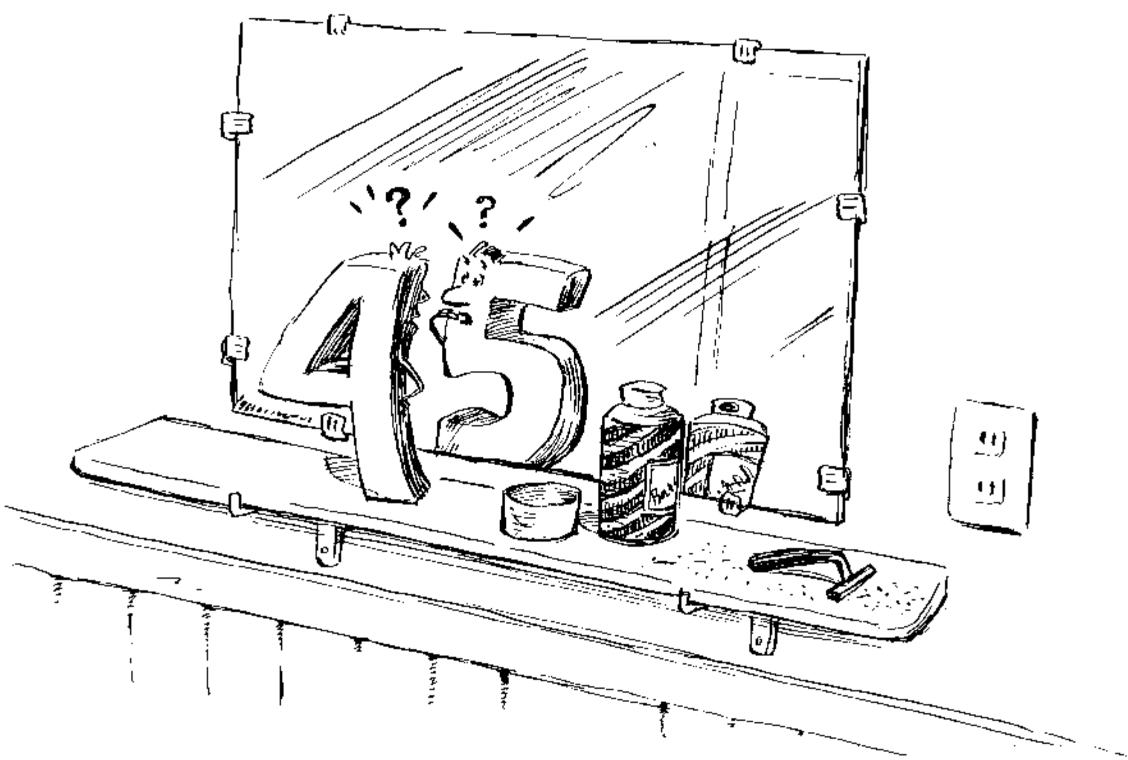
$$\Rightarrow \quad 4^2 - 2 \cdot 4 \cdot \frac{9}{2} + \left(\frac{9}{2}\right)^2 = 5^2 - 2 \cdot 5 \cdot \frac{9}{2} + \left(\frac{9}{2}\right)^2 \tag{3}$$

$$\Rightarrow \quad \left(4 - \frac{9}{2}\right)^2 = \left(5 - \frac{9}{2}\right)^2 \tag{4}$$

$$\Rightarrow \quad 4 - \frac{9}{2} = 5 - \frac{9}{2} \tag{5}$$

$$\Rightarrow \quad 4 = 5 \tag{6}$$

The reasoning used to reach the conclusion that $4 = 5$ began with a statement that is obviously true ($16 - 36$ and $25 - 45$ are both equal to -20) and ended with a statement that is obviously false ($4 = 5$). What went wrong?



2
Four Equals Five

KEY CONCEPTS
- Equations
- Equations, quadratic
- Identities
- Solutions, extraneous
- Square roots

Explanation

All the steps in the reasoning are correct except the step from equation 4 to equation 5:

$$\left(4-\frac{9}{2}\right)^2=\left(5-\frac{9}{2}\right)^2 \Rightarrow 4-\frac{9}{2}=5-\frac{9}{2}$$

In general $a^2=b^2 \Rightarrow a=b$ or $a=-b$, as the following algebraic reasoning shows:

$$a^2=b^2 \Rightarrow a^2-b^2=0 \Rightarrow (a-b)(a+b)=0$$

$$\Rightarrow a-b=0 \text{ or } a+b=0 \Rightarrow a=b \text{ or } a=-b$$

So by correctly deducing from $\left(4-\frac{9}{2}\right)^2=\left(5-\frac{9}{2}\right)^2$ that *either* $4-\frac{9}{2}=5-\frac{9}{2}$ (false) *or* $4-\frac{9}{2}=-\left(5-\frac{9}{2}\right)$ (true), the paradox disappears.

Comments

- It is common for students to write $\sqrt{x^2}=x$, and that is the source of the confusion caused by this paradox. A correct statement is $\sqrt{x^2}=|x|$, which is how the modulus, or absolute value, function is sometimes defined.
- The incorrect statement $\sqrt{x^2}=x$ (which is false if x is negative) is often written $(x^2)^{1/2}=x$. Writing the statement in this way is an example of misusing the index laws. It is true that $(a^m)^n=a^{mn}$ for integer values of m and n and for any value of a, both positive and negative. It is also true that $(a^m)^n=a^{mn}$ if a is positive and m and n are real. However, this statement may *not* be true if a is negative and m and n are not integers.
- The square root notation ($\sqrt{\ }$) gives some difficulty to students. It needs to be stressed that $\sqrt{a}$ is defined as the positive square root of a and is not a two-valued concept. Thus, it is correct to write $\sqrt{9}=3$ and $\sqrt{9}=-3$, but it is not correct to write $\sqrt{9}=\pm 3$. (The correct notation is $\pm\sqrt{9}=\pm 3$.)
- The confusion about the square root notation probably originates with the language we use regarding the notation. It is correct to state that 9 has two square roots, +3 and −3, because these are the two solutions to the equation $x^2=9$. But although it is incorrect to write $\sqrt{9}=-3$, the notation $\sqrt{9}$ is often read as "the square root of 9" or just "root 9," which students often confuse with stating that −3 is a square root of 9. Similar comments apply to the notation $9^{1/2}$, which means exactly the same thing as $\sqrt{9}$.

- In the same vein, it is interesting that some students will write $\sqrt{9} = \pm 3$ but will regard $\sqrt{2}$ as a positive number, not as a pair of numbers. Additionally, the confusion in students' minds about the meaning of the square root notation is probably reinforced by the term $\sqrt{b^2 - 4ac}$, which occurs in the formula for the solution of the quadratic equation.
- The "proof" that $4 = 5$ can be generalized to "prove" that $x = x + 1$ for all x:

  Begin with $x^2 - x(2x+1) = (x+1)^2 - (x+1)(2x+1)$ (which is true, because both sides are equal to $-x^2 - x$).

Complete the square on both sides by adding $\left[\frac{1}{2}(2x+1)\right]^2$:

$$x^2 - x(2x+1) + \left[\frac{1}{2}(2x+1)\right]^2 = (x+1)^2 - (x+1)(2x+1) + \left[\frac{1}{2}(2x+1)\right]^2$$

$$\Rightarrow \quad \left[x - \frac{1}{2}(2x+1)\right]^2 = \left[(x+1) - \frac{1}{2}(2x+1)\right]^2$$

$$\Rightarrow \quad \left[x - \frac{1}{2}(2x+1)\right] = (x+1) - \frac{1}{2}(2x+1)$$

$$\Rightarrow \quad x = x + 1$$

In particular, this argument apparently proves that any two consecutive integers are equal!

See Also

For more about index laws, see Activity 6.