3 Geometry, Spatial Reasoning, and Measurement

Geometry, spatial reasoning, and measurement are essential areas in children's development. Children learn about geometric shapes and structures as important topics in themselves. But these topics also support children's learning of other mathematics, such as number, arithmetic, and patterns. Spatial reasoning complements geometric knowledge. Spatial reasoning includes spatial orientation—knowing how to get around in the world—and spatial visualization—knowing how to build and manipulative objects mentally, including composing and decomposing objects. Geometric measurement rounds out the core components. *Geometry* means "earth measure," and geometry, spatial reasoning, and measurement are topics that connect to each other and to other mathematics and that connect mathematics to real-world situations. For example, these core components are the foundations of number lines, arrays in multiplication, fractions, graphing, and topics beyond. They also lie at the heart of physics, chemistry, biology, geology and geography, and art and architecture. So, across many areas, we see that a picture—or diagram or figure—can be "worth a thousand words." Increasing students' sophistication with visualization and imagery increases the meaning they can take from that "picture," including those seen daily on television, in video games, on GPS maps, and so forth.

Unfortunately, geometry and measurement are two of U.S. students' weakest topics in mathematics. Even in kindergarten, children in the U.S. know less about shape than children in other countries. Fortunately, they know enough to build on, they can learn a lot quickly, and they enjoy engaging with shapes, space, and measurements. Indeed, young children *play* with shapes and geometric structures naturally. In a study of the mathematics that children engage in spontaneously in their play, the most frequent topic was shapes and structures.

Geometry

From the earliest years, children learn about shape and use shapes to learn. In learning the *geometry* of shapes, they progress through increasingly powerful levels of thinking about shapes. For example, at first, they cannot explicitly distinguish circles, triangles, and squares from nonexamples. They gradually develop richer visual templates for these categories and eventually learn about the parts and attributes of the shapes. This is especially important if they did not receive high-quality geometric experiences in preschool, because shape research suggests that concepts can become inflexible by the end of kindergarten or first grade.

Table 3.1 presents the developmental progressions for ideas and skills for geometry and spatial reasoning.

Table 3.1

Progression of Ideas and Skills for Geometry, Spatial Reasoning, and Measurement

Prekindergarten	Kindergarten	Grade 1
Shape and Structure	**Shape and Structure**	**Shape and Structure**
2s/3s: Recognize two-dimensional shapes informally (including at least circles, squares, then triangles, rectangles) in different orientations. Discriminate between two-dimensional and three-dimensional shapes intuitively, marked by accurate matching or naming. See and describe pictures of objects (e.g., recognize a three-dimensional object on a two-dimensional page of a book). 4s/pre-Ks: Recognize and describe two-dimensional shapes regardless of orientation, size, and shape (including circles and half-/quarter-circles, squares and rectangles, triangles, and regular rhombi, trapezoids, hexagons). Describe shapes by number of sides and/or corners (up to the number they can count) and sides of same or different length. Describe the difference between two-dimensional and three-dimensional shapes, and name common three-dimensional shapes informally or with mathematical names ("ball"/sphere; "box" or rectangular prism, "rectangular block" or "triangular block"; "can"/ cylinder).	Recognize and describe a wide variety of two-dimensional shapes (e.g., octagons, parallelograms, convex/concave figures) regardless of orientation, size, and shape. Sort shapes by number of sides and/or corners and length relationships between sides. Recognize and name common three-dimensional shapes (including real-world objects), including spheres, cylinders, [rectangular] prisms, and pyramids.	Name most common shapes, including rhombuses, without making such mistakes as calling ovals "circles." Recognize (at least) right angles, so distinguishing between a rectangle and a parallelogram without right angles. Use manipulatives representing parts of shapes, such as sides and angle "connectors," to make a shape that is completely correct on the basis of knowledge of components and relationships.
Spatial Relations	**Spatial Relations**	**Spatial Relations**
2s/3s: Enact spatial movements informally using such relational terms as *up, down, on, off,* and *under.* 4s/pre-Ks: Match shapes by intuitively using geometric motions to superimpose them. Use relational words of proximity, such as *beside, next to,* and *between,* referring to a two-dimensional environment. Match the faces of three-dimensional shapes to two-dimensional shapes, naming the two-dimensional shapes.	Begin to use relational language of *right* and *left.* Identify and create symmetric figures (e.g., mirrors as reflections).	Use geometric motions to create symmetric figures (e.g., paper folding; also mirrors as reflections) and determine congruence.
Compositions and Decompositions in Space	**Compositions and Decompositions in Space**	**Compositions and Decompositions in Space**
2s/3s: Solve simple puzzles involving things in the world. Create pictures by representing single objects, each with a different shape. Combine unit blocks by stacking. 4s/pre-Ks: Move shapes using slides and flips, and turn them to combine shapes to build pictures. Copy a design shown on a grid, placing squares and rectangles onto squared-grid paper. Combine building blocks using multiple spatial relations to produce composite shapes (arches, enclosures, corners, and crosses).	Create pattern-block designs (those with multiples of 60-degree and 120-degree angles). Create compositions and complete puzzles with systematicity and anticipation, using a variety of shape sets (e.g., pattern blocks; rectangular grids with squares, right triangles, and rectangles; tangrams). Build simple three-dimensional structures from pictured models.	Make new two-dimensional shapes, shape structures out of smaller shapes, and substitute groups of shapes for other shapes to create new shapes in different ways. (See related area goals.)

Table 3.1—*Continued*

Prekindergarten	Kindergarten	Grade 1
Concept of Measurement	**Concept of Measurement**	**Concept of Measurement**
2s/3s: Identify two-dimensional and three-dimensional objects as "the same" or "different" in size. 4s/pre-Ks: Identify objects and drawings as "more" or "less" on the basis of attributes they can identify (and later can measure), such as length and area, and solve problems by making direct comparisons of objects on the basis of those attributes.	Use measurable attributes, such as length or area, to solve problems by comparing and ordering objects.	Compose and decompose plane and solid shapes, thus building an understanding of part-whole relationships and developing the background for working with units composed of units. (These relate to the geometry goals.)
Length	**Length**	**Length**
2s/3s: Intuitively recognize length as extent of one-dimensional space.Compare two objects directly, noting equality or inequality. 4s/pre-Ks: Begin to measure by laying units end to end. Understand that lengths can be concatenated to make a new length.	Compare the lengths of two objects both directly (by comparing them with each other) and indirectly (by comparing both with a third object), and order several objects according to length (even if differences between consecutive lengths are small). Measure by laying units end to end, covering the whole without gaps, and count the units to find the total length.	Measure by repeated use of a unit, and apply the resulting measures to comparison situations.
Area	**Area**	**Area**
2s/3s: Use side-matching strategies in comparing areas. 4s/pre-Ks: Compare areas for tasks that suggest superposition or show decomposition into squares.	Cover a rectangular region with square units. Count squares in rectangular arrays correctly and (increasingly) systematically.	Make and draw coverings of simple rectangular regions with square units. For rectangles two squares high or wide, count the rows or columns of two by twos.
Volume	**Volume**	**Volume**
2s/3s: Identify capacity or volume as attribute. 4s/pre-Ks: Compare two containers directly by pouring.	Compare two containers using a third container and (at least implicitly) transitive reasoning. Fill rectangular containers with cubes and/or make rectangular prisms ("buildings") from layers of blocks.	Fill rectangular containers with cubes, completing one layer at a time with cubes, and/or make rectangular prisms ("buildings") from layers of blocks.

Note: Some compositions and decompositions activities overlap with measurement (area and volume). Grade 3 develops the area work above as a setting for multiplication. Fuller development of area is a grade 4 Focal Point, and fuller development of volume is a grade 5 Focal Point.

Shape and Structure

Kindergartners form visual templates, or models of shape categories. For example, children recognize a shape as a rectangle because "it looks like a door." Because children base their understanding of shapes on examples, they need to experience a rich variety of shapes in each shape category so that their mental models are not overly restricted. For example, children without good experiences often reject both triangles and rectangles that are "too skinny" or "not wide enough."

Children should see examples of rectangles that are long and skinny, and they should contrast rectangles with nonrectangles that appear similar but do not have an important defining attribute (see fig. 3.1). Similarly, they should see examples of triangles that have sides of three different lengths, and they should contrast triangles with nontriangles (see fig. 3.2).

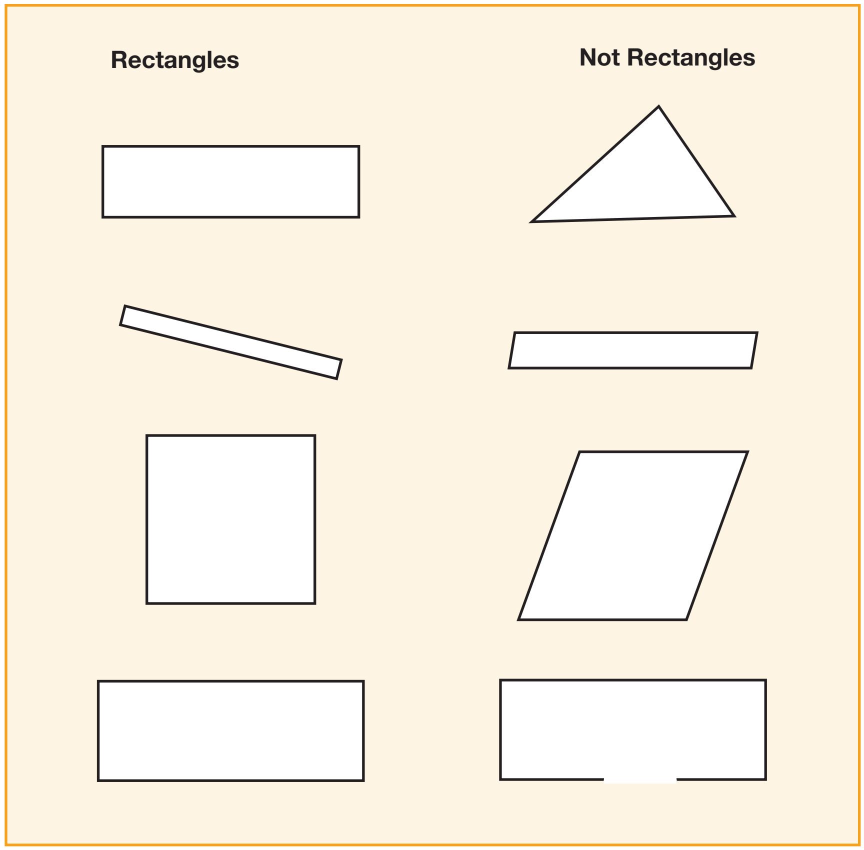


Fig. 3.1. Rectangles and nonrectangles

Children also need to see examples of shapes beyond circles, squares, rectangles, and triangles. Without these, children develop limited notions. For example, many children come to believe incorrectly that a geometric figure such as a trapezoid "is not a shape" because it is not a shape for which they know a name (and many know only "circle," "square," "triangle," and "rectangle"). Kindergartners can learn to recognize not only trapezoids but such shapes as rhombuses, parallelograms, and octagons. Figure 3.3 provides information for the teacher about several two-dimensional geometric shapes. The term "two-dimensional shapes" means flat shapes (e.g., plane shapes that could be drawn in their entirety on paper) that are *closed*, meaning that they have no "loose, dangling ends," and are *connected*, meaning that they are in one single piece and do not have sides that cross each other.

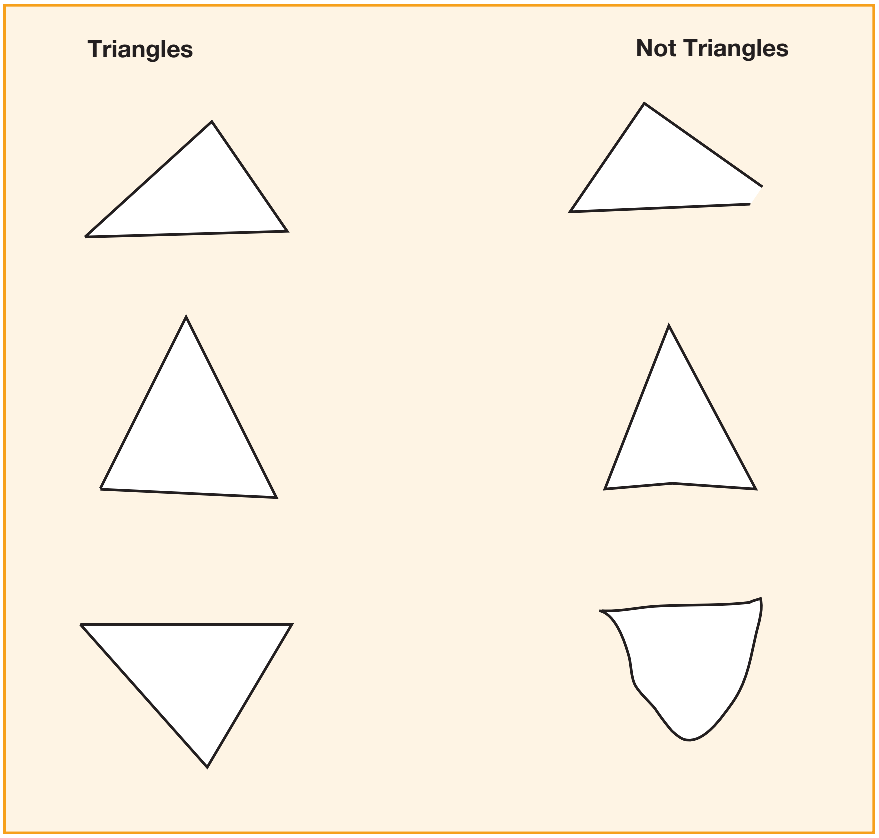


Fig. 3.2. Triangles and nontriangles

Kindergartners should also learn to recognize these shapes whether they are in "standard position" or rotated so that their bases are not horizontal. Although it is especially important to provide varied examples for children who have not had a good preschool experience, development of flexible, accurate "visual thinking" should continue throughout children's education, even as more mathematically explicit and sophisticated levels of thinking and language take precedence.

Kindergartners can begin to develop such explicit and sophisticated levels of thinking and communication. They can learn to describe, and even define, shapes in terms of their *parts* or *attributes* (properties). For example, they can build accurate representations of shapes from physical models of line segments, such as sticks. As they discuss what they have built, attributes of the shapes will arise naturally. That is, they may say that what they build is a rectangle *because* it has two pairs of sides that are equal in length and all right angles. This experience of discussing attributes of rectangles helps children begin to understand the *geometric structure* of all rectangles at an explicit level of thinking.

Similarly, when guided by the teacher, children notice symmetry not only in shapes, such as rectangles, but in their environment. They begin to design and extend symmetry into their block buildings and artwork.

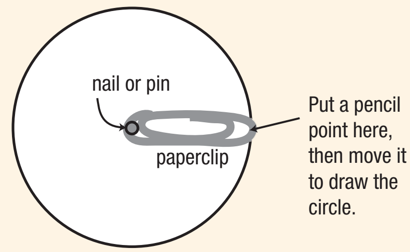
Definitions of Familiar Two-Dimensional Shapes

Triangles are those two-dimensional shapes that have three straight sides (see fig. 3.2).

Rectangles are those two-dimensional shapes that have four straight sides and four right angles (see fig. 3.1). Corners of standard pieces of paper are usually (approximately) right angles. You can make right angles using doublefolding (see the sidebar box "What Are Right Angles?").

Squares are those two-dimensional shapes that have four straight sides of the same length and have four right angles. Notice that any square is also a rectangle because it has four straight sides and four right angles. So squares could be defined as special rectangles that have all sides the same length.

Informally, circles are those two-dimensional shapes that are "perfectly round." From a mathematical perspective, however, circle are those two-dimensional shapes that consist of all points that are a fixed distance from a fixed center point. Any simple tool (such as a compass or even a pencil tied to a fixed length of string) that holds a pencil point a fixed distance away from a center point will draw a circle, as shown here.

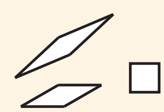


A circle consists of all the locations that are a fixed distance (here: a paperclip length) away from a fixed point (here: the nail).

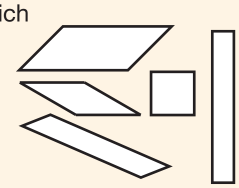
Definitions of Other Two-Dimensional Shapes

Quadrilaterals are those two-dimensional shapes that have four straight sides. Notice that squares and rectangles (as well as rhombuses, parallelograms, and trapezoids) are also quadrilaterals because these shapes have four straight sides.

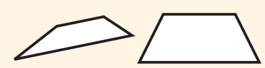
Rhombuses are those two-dimensional shapes that have four straight sides of the same length. Notice that every square is also a rhombus because it has four sides of the same length.



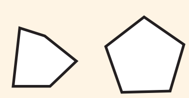
Parallelograms are those two-dimensional shapes that have four straight sides and for which each pair of opposite sides are parallel. Informally, two straight sides are parallel if it is possible to slide one without turning so that both lie on the same straight line. Or two are parallel if when both are extended to become infinitely long straight lines, they never meet.



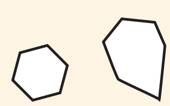
Trapezoids are those two-dimensional shapes that have four straight sides and at least one pair of parallel sides. (Some people define trapezoids as those two-dimensional shapes that have four straight sides and exactly one pair of parallel sides.)



Pentagons are those two-dimensional shapes that have five straight sides.



Hexagons are those two-dimensional shapes that have six straight sides.



Octagons are those two-dimensional shapes that have eight straight sides.

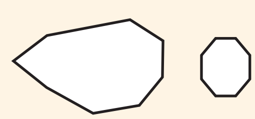
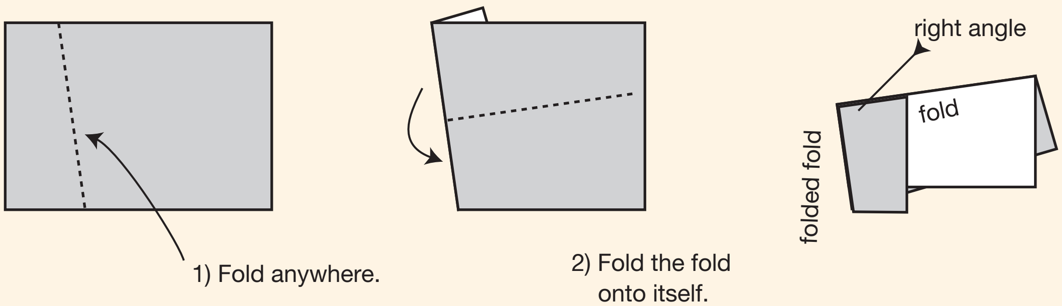


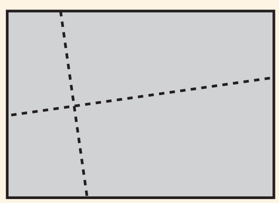
Fig. 3.3. Familiar two-dimensional shapes

What Are Right Angles?

Four right angles fit snugly together.

We can create right angles by folding paper:





The fold lines meet at right angles.

Another valuable activity is the tactile-kinesthetic exploration of shapes—feeling shapes hidden in a box. Even if previously done in preschool, this activity can grow with children. Kindergartners can name the shape they are feeling rather than just match shapes. After this, they can extend the activity further as they *describe* the shape without using its name, so that their friends can name the shape. In this way, children learn the properties of the shape, moving from intuitive to explicit, verbalized knowledge. All these variations can be repeated with less familiar shapes.

Such activities help children learn to identify and describe shapes by the number of their sides or corners, as illustrated by a kindergartner who declared that an obtuse ("long and skinny") triangle "*must* be a triangle because it has three sides." Such descriptions build geometric concepts but also reasoning skills and language. They encourage children to view shapes analytically. Children begin to describe some shapes in terms of their properties, such as saying that squares have four sides of *equal length*. They informally describe properties of blocks in functional contexts, such as that some blocks roll and others do not.