3 Focusing on Fractions and Decimals

In grade 4, students are introduced to decimals. The focus in this grade is on understanding relationships between fractions and decimals. Like fractions, numbers with decimal places to the right of the ones place can be used to name values that are between whole numbers. To visualize fraction-decimal relationships, students use fraction models with which they are familiar, especially area models and the number line. An emphasis is placed on the understanding that decimal notation is an extension to the right of the ones place of the base-ten system. Students use their understanding of relationships between fractions and decimals as well as decimal place value to visualize, represent, and communicate about decimals.

Instructional Progression for Connecting Fractions and Decimals

The focus on fluency with fractions and decimals in grade 4 is supported by a progression of related mathematical ideas before and after grade 4, as shown in table 3.1. To give perspective to the grade 4 work, we first discuss some of the important ideas that students focused on in grade 3 that prepared them for expanded work with fractions and new work with decimals in grade 4. At the end of the detailed discussion of this grade 4 focal point, we present examples of how students will use their fraction and decimal understandings and skills in later grades. For more detailed discussions of the "before" and "after" parts of the instructional progression, please see the appropriate grade-level books, that is, Focus in Grade 3 (NCTM 2009) and Focus in Grade 5 (NCTM 2009).

Table 3.1 represents an instructional progression for the understanding of fractions and decimals in grades 3 through 5.

Early Foundations of Fractions

In grade 3, students begin their focus on fractions. In the study of fractions, a variety of concrete and pictorial experiences have students represent a fraction as a part of a whole in an area model, as a part of a set, and as a distance designated by a point on the number line (a type of linear model). Students begin to build understanding of the idea that a fraction is a relationship of two numbers—the denominator, which names the parts on the basis of how many equal parts are in the whole, and the numerator, which tells the number of equal parts being considered.

To develop an understanding of a fraction as part of a whole, students can use area models. For example, the area models in figure 3.1 show 3 one-fourths, or 3/4.

Table 3.1
Grade 4: Focusing on Fractions and Decimals
Instructional Progression for Developing Understanding in Grades 3–5

Grade 3	Grade 4	Grade 5
Students use unit fractions (1/n) to represent equal divisions of a whole. Students create nonunit fractions by joining unit fractions (e.g. 2/3 is the same as 1/3 + 1/3) and build a whole (1) by joining "n" of the unit fraction (1/n). Students judge the size of a fractional part by relating it to the size of the whole. Students compare unit fractions of the same-sized whole by observing that the larger the denominator, the smaller the amount represented by the unit fraction. Students use fractions to represent numbers that are equal to, less than, or greater than 1. Students compare and order fractions by using models, benchmark fractions, common numerators, and common denominators. Students use models, including the number line, to identify equivalent fractions. Students use fractional parts of units to measure length. **	Students analyze techniques that involve multiplication and division to generate equivalent fractions.** Students identify equivalent symbolic representations of improper fractions and mixed numbers. Students use decimal notation as an extension of the base-ten system to the right of the ones place. Students use their understanding of fractions and place value to read and write decimals that are greater than 1 or between 0 and 1. Students connect equivalent fractions and decimals by comparing models with symbols and using equivalent symbols to describe the same point on a number line. Students use place-value notation and equivalent fractions to identify equivalent decimals. Students use place-value notation and understanding of fractions to compare and order decimals.	Students apply their understanding of fractions and fraction models to add and subtract fractions with like and unlike denominators. Students apply their understanding of decimal models, place value, and properties to add and subtract decimals. Students work toward fluency in adding and subtracting fractions and decimals. Students make reasonable estimates of fraction and decimal sums and differences. Students add and subtract fractions and decimals to solve problems, including problems involving measurement.

* Appears in Grade 3 Connections to the Focal Points.

** Appears in Grade 4 Connections to the Focal Points.

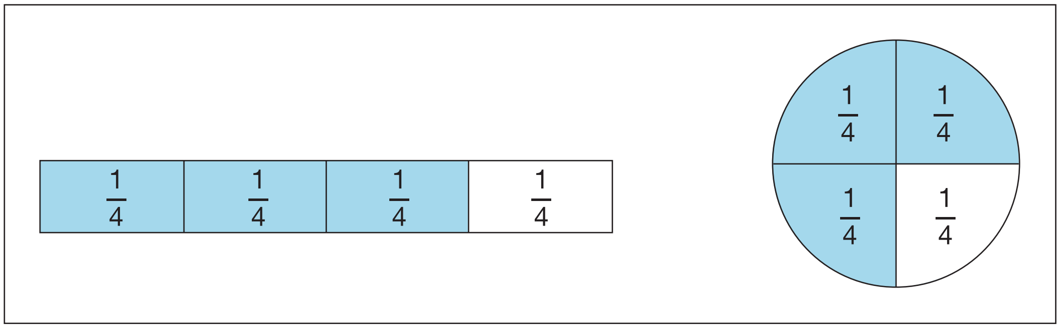

Fig. 3.1. Using area models to show 3/4

Students also use linear models, such as fraction strips, fraction bars, and the number line, to represent a fraction as a part of a whole. For example, the fraction-bar model in figure 3.2 shows 3/8.

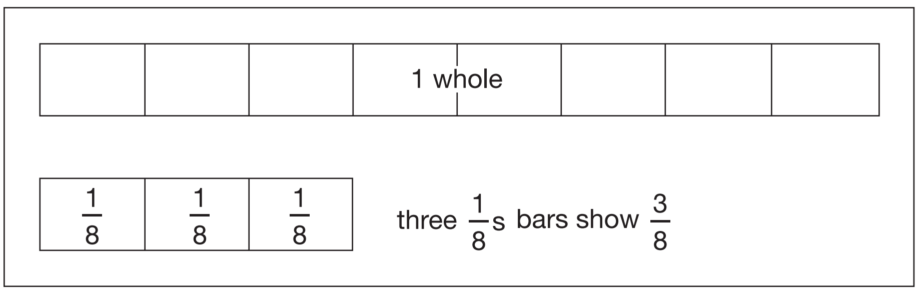

Fig. 3.2. Using fraction bars to show 3/8

Students can also illustrate the same relationship between a fraction (e.g., 3/8) and the unit (1) by using a point on the number line, as in figure 3.3.

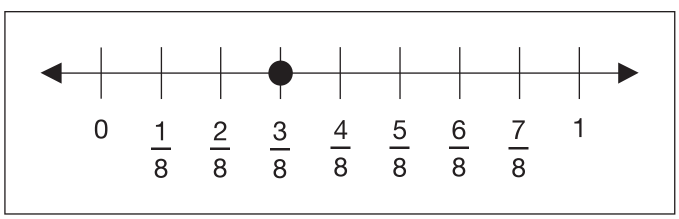

Fig. 3.3. Using the number line to represent 3/8

To show that a fraction can identify a part of a whole set, students can use set models like the model shown in figure 3.4.

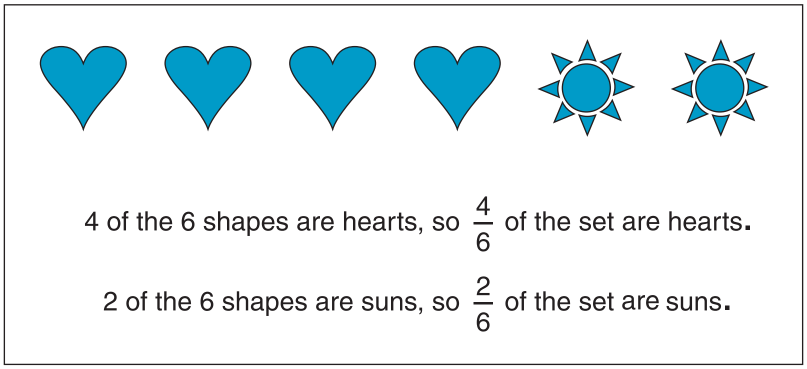

Fig. 3.4. Representing a fraction as a part of a set

Students can use these same types of representations to learn about fractions equal to, or greater than, 1. Students use models like the one shown in figure 3.5 to show the relationship between improper fractions (fractions in which the numerator is equal to, or greater than, the denominator) and whole numbers or mixed numbers. Students can see that 7 thirds are shaded, so the model shows 7/3. They also see that 2 wholes and 1/3 of another whole are shaded, so the model shows $2\frac{1}{3}$. From this model, students learn that $\frac{7}{3} = 2\frac{1}{3}$. As students work with these types of models, they may also notice that some improper fractions are equivalent to whole numbers rather than mixed numbers. For example, in figure 3.5, they can also see that 6/3 = 2.

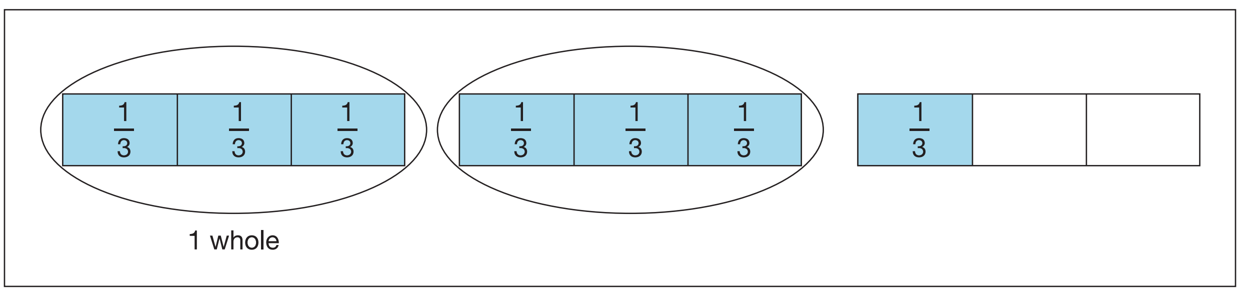

Fig. 3.5. Using a model to establish the equivalence between $\frac{7}{3}$ and $2\frac{1}{3}$

In grade 3, students have opportunities to see that different fractions are equivalent if they represent the same amount in relation to the whole. Students can use area and linear models as shown in figure 3.6 to visualize equivalent fractions. The models show that 1/3 = 2/6 = 3/9 and that 2/3 = 4/6 = 6/9 = 8/12.

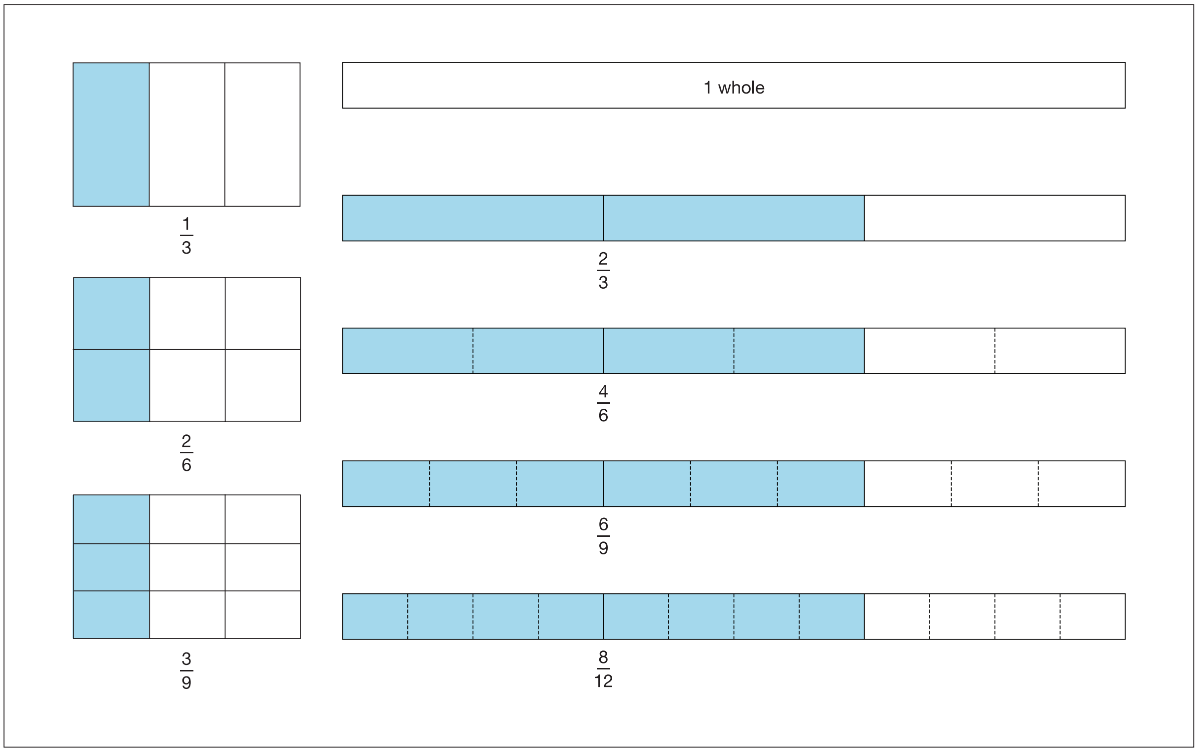

Fig. 3.6. Area and linear models representing equivalent fractions

Models such as the ones in figure 3.6 also emphasize the relationship between the number of parts in the whole and the size of each part. Students can use these models to internalize the ideas that the more parts into which a whole is divided, the smaller the size of each part, and the more parts that are needed to form the same portion of the whole. These important relationships foster the understanding that students will call on as they learn to use multiplication and division to find equivalent fractions in grade 4, as well as when they add and subtract fractions with unlike denominators in grade 5.

In grade 3, students are also asked to compare two fractions that are not equal. To compare fractions that have the same denominator, students can use their understanding that two fractions with the same denominator indicate a whole or wholes that have been divided into the same number of equal-sized parts, so the fraction with the larger numerator has the larger number of equal parts and is the larger fraction. For example, the number-line model in figure 3.7 clearly shows that 5/8 is larger than 3/8.

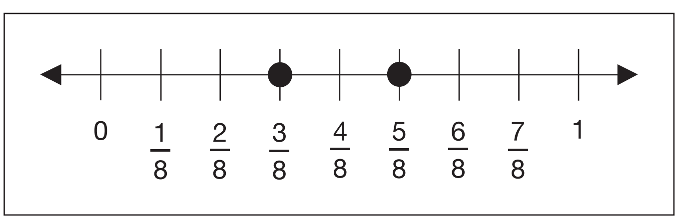

Fig. 3.7. Number-line representation of the principle that if the denominators are the same, the fraction with the larger numerator is the larger fraction

To compare fractions that have the same numerator, students can draw on the understanding of the principle of the inverse relationship between the number of equal parts in a whole and the size of the parts. For example, as the models for 3/8 and 3/10 presented in figure 3.8 illustrate, in relation to the same whole, tenths are smaller than eighths, so 3/10 is smaller than 3/8.

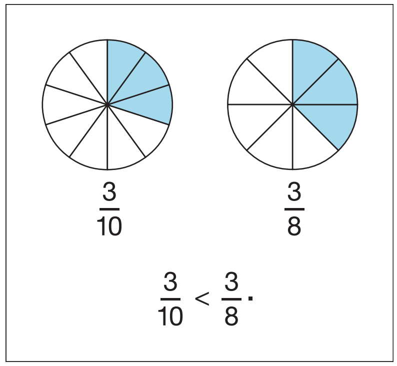

Fig. 3.8. Pictorial representation of the principle that if the numerators are the same, the fraction with the larger denominator is the smaller fraction

Students can also use fractional number sense to compare fractions by considering how they relate to benchmark numbers, such as 0, 1/2, or 1. For example, if a student knows that one fraction is less than 1/2 and another fraction is greater than 1/2, then the student can determine that the first fraction is less than the second fraction. Students can also compare fractions with 1. For example, to compare 4/5 and 7/8, students understand that if 4/5 is 1/5 away from 1 whereas 7/8 is 1/8 away from 1, and that 1/8 is less than 1/5, then 7/8 is closer to 1 than 4/5 is. So 7/8 > 4/5.

In grade 3, students can also use the fractional number sense they have developed to order fractions. For example, students might order 2/5, 1/5, and 5/7 by reasoning that if 2/5 is less than 1/2 whereas 5/7 is greater than 1/2, and that 1/5 is less than 2/5, then the order of the numbers is 1/5, 2/5, 5/7.

In grade 4, students will augment these strategies for comparing and ordering fractions by learning how to use multiplication and division to write equivalent fractions with the same denominator to compare fractions. For further details on the grade 3 focal point Focusing on Fractions, refer to Focus in Grade 3 (NCTM 2009).

Connecting Fractions and Decimals

Extending fraction concepts

In grade 4, students continue to connect the concrete and pictorial representations used in grade 3 with symbolic representations to extend fraction concepts. Finding patterns in equivalent fractions is one example of this extension. In grade 4, students begin to use multiplication and division to identify the equivalent fractions that they recognized with models in grade 3.

In grade 4, students are led to the generalization that to find a fraction equivalent to another fraction, they can multiply or divide the numerator and denominator by the same number. The fraction strip becomes a powerful representation through which students can gain an understanding of this mathematical relationship. For example, as shown in figure 3.9, students can fold a fraction strip into thirds and shade 2 of the thirds to show 2/3. Then they can fold the strip again to divide each third in half. As students interpret the fraction strip, they can see that their strip shows sixths and that what was 2 thirds is now

4 sixths, or 2/3 = 4/6. Students begin to understand, then, the fundamental principle of using multiplication to find fractions. When they folded each third in half, they multiplied the number of parts in the whole, that is, the denominator, 3, by 2. When they counted the parts, 2 times as many parts were shaded, which is the same as multiplying the numerator, 2, by 2. As students' understanding matures, they can use the models to internalize the crucial relationship between the size of the parts and the number of parts in the whole that they developed in grade 3. They begin to understand that 2/3 and 4/6 represent the same amount of the strip because, although the whole has 2 times as many parts in 4/6 than it does in 2/3, the parts are half as big. So multiplying the numerator and denominator by 2 results in an equivalent fraction that names the same part of the whole. Students can apply the same understanding to multiply the numerator and denominator by 3 to find that 2/3 = 6/9, and multiply the numerator and denominator by 4 to find that 2/3 = 8/12.

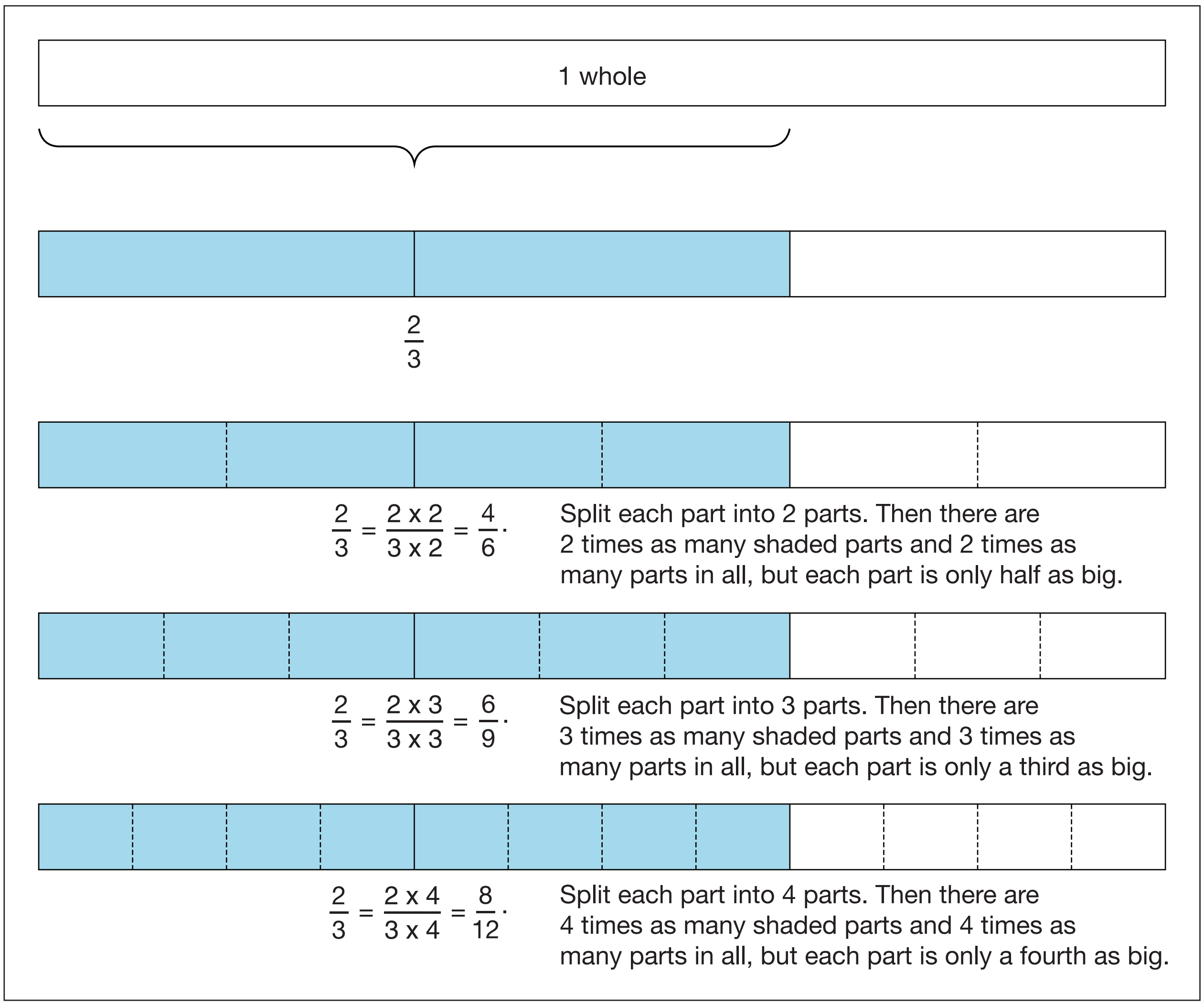

Fig. 3.9. Linear model illustrating why forming equivalent fractions by splitting parts is the same as multiplying the total number of parts and the number of shaded parts by the same number

An interesting relationship to note in figure 3.9 is that students physically divide the whole fraction bar, yet numerically they multiply the numerator and denominator of 2/3. As students interpret a variety of fraction models such as the ones shown in figure 3.9, they are given opportunities to observe that when the denominator and numerator of a fraction are each multiplied by a positive number, the resulting fraction refers to more parts but each part is a smaller fraction of a whole.

By reversing the process shown in figure 3.9 and joining the parts of a whole, students can also use a fraction strip to demonstrate why dividing the numerator and denominator of a fraction by the same number results in an equivalent fraction. As shown in figure 3.10, the whole is divided into 15 equal parts and 10 of them are shaded to model 10/15. Students can join groups of 5 parts. The result is a whole that is divided into 3 equal groups. As students interpret the model, they can see that the strip now shows thirds and that what was 10 fifteenths is now 2 thirds, so 10/15 = 2/3.

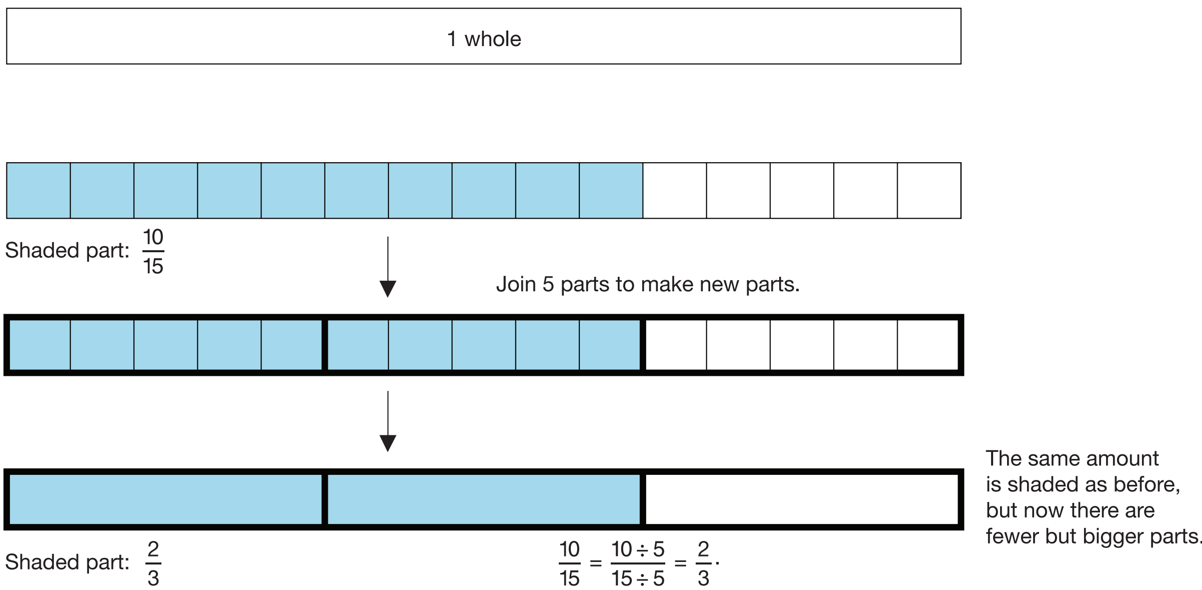

Fig. 3.10. Linear model illustrating how forming equivalent fractions by joining original parts into larger parts is related to forming equivalent fractions by separating the original parts into smaller parts

Students begin to understand, then, the fundamental principle of using division of both the numerator and denominator to find equivalent fractions. When they joined groups of 5 parts, they divided the number of parts in the whole, that is, the denominator, 15, by 5. When they counted the parts, 2 parts were shaded, which illustrated that they divided the numerator, 10, by 5. This pattern further emphasizes the important relationship between the size of the parts and the number of parts in the whole. The fractions 10/15 and 2/3 represent the same amount of the strip because, although the whole is partitioned into one-fifth as many parts in 2/3 as in 10/15, the parts are 5 times larger. So dividing each of the numerator and denominator of 10/15 by 5 results in an equivalent fraction.

In figure 3.9, the number of parts was multiplied by a number resulting in more parts. In figure 3.10, the number of parts was divided by a number resulting in fewer parts. Through an abundance of concrete and pictorial experiences with separating and joining groups to make related representations of equal fractional amounts, students experience that multiplying or dividing the numerator and denominator by the same number results in equivalent fractions.

Students can use a multiplication table to apply the multiplicative patterns in equivalent fractions. For example, in the hundreds chart in figure 3.11, students can look from left to right across the 2 row for numerators and across the 5 row for denominators to see that 2/5 = 4/10 = 6/15 = 8/20 = 10/25, and so on. They can also look from right to left across the 4 row for numerators and the 9 row for denominators to see that 4/5 = 40/90 = 36/81 = 32/72 = 28/63 = 24/54, and so on. As students' understanding of equivalent fractions matures, they can be led to observe that this pattern occurs because the numbers in each row increase by a factor of 2 from column 1 to column 2, by a factor of 3 from column 1 to column 3, by a factor of 4 from column 1 to column 4, and so on, as indicated by the equations in figure 3.11.

x	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

$$\underset{\times 2}{\overset{\times 2}{\frac{2}{5} = \frac{4}{10}}} \qquad \underset{\times 3}{\overset{\times 3}{\frac{2}{5} = \frac{6}{15}}} \qquad \underset{\times 4}{\overset{\times 4}{\frac{2}{5} = \frac{8}{20}}} \qquad \underset{\times 5}{\overset{\times 5}{\frac{2}{5} = \frac{10}{25}}} \qquad \underset{\times 6}{\overset{\times 6}{\frac{2}{5} = \frac{12}{30}}}$$

$$\underset{\div 10}{\overset{\div 10}{\frac{40}{90} = \frac{4}{9}}} \qquad \underset{\div 9}{\overset{\div 9}{\frac{36}{81} = \frac{4}{9}}} \qquad \underset{\div 8}{\overset{\div 8}{\frac{32}{72} = \frac{4}{9}}} \qquad \underset{\div 7}{\overset{\div 7}{\frac{28}{63} = \frac{4}{9}}} \qquad \underset{\div 6}{\overset{\div 6}{\frac{24}{54} = \frac{4}{9}}}$$

Fig 3.11. Equivalent fractions in the multiplication table

The ability to symbolically calculate equivalent fractions is essential to students' further study of fractions. They can use this method when they compare fractions. For example, they may have difficulty using benchmarks or fractional number sense to compare such fractions as 4/7 and 3/5. So to compare such fractions without models, students can find equivalent fractions with the same denominators. They can use the process described above to multiply the numerator and denominator in 4/7 by 5 to find that 4/7 =

20/35 and to multiply the numerator and denominator in 3/5 by 7 to find that 3/5 = 21/35. Once the fractions are written with like denominators, students can compare them by comparing numerators: 20/35 < 21/35, so 4/7 < 3/5. In grade 5, students will use equivalent fractions with like denominators to add and subtract any pair of fractions.

The relationship between improper fractions and mixed numbers

In grade 4, students work on transitioning from the concrete and pictorial to the symbolic as they extend their understanding of relationships between improper fractions and mixed numbers. Students formalize this relationship when they learn how to write an improper fraction as a mixed number and a mixed number as an improper fraction without the use of the models that formed the basis for their work in grade 3. In grade 3, students observed that numbers greater than 1 that are not whole numbers could be written as equivalent improper fractions and mixed numbers. For example, the model in 3.12 shows 14/3 because 14 thirds are shaded; as well as shows $4\frac{2}{3}$ because 4 wholes and $\frac{2}{3}$ of another whole are shaded.

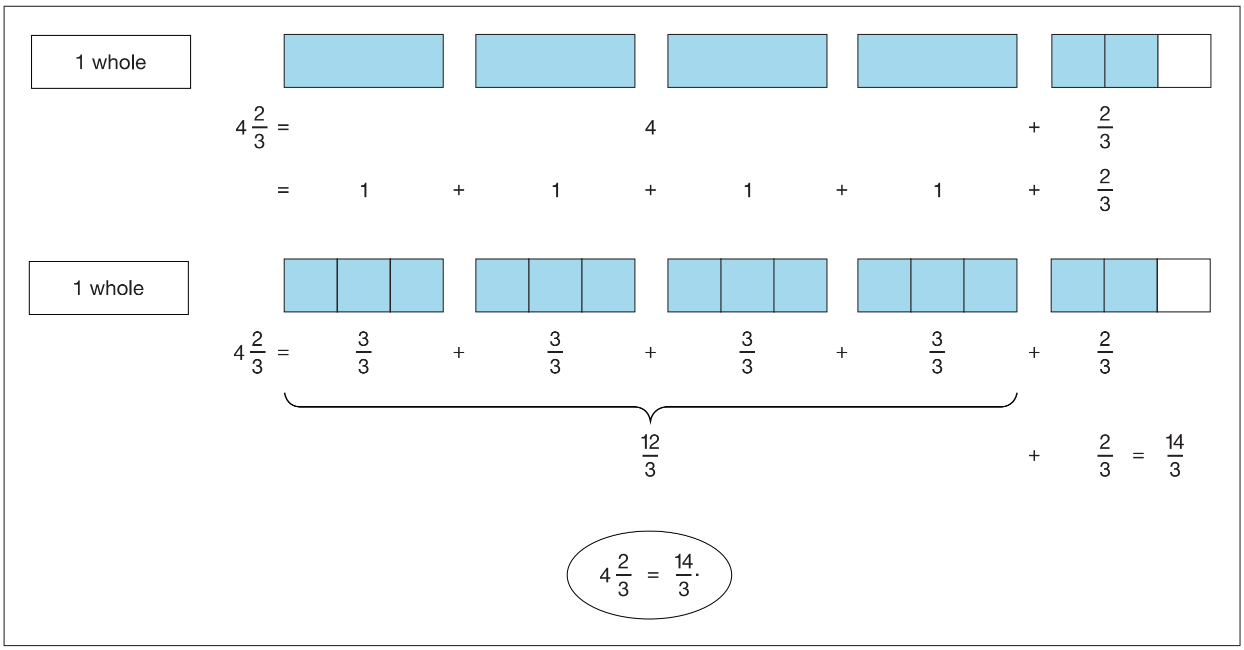

Fig. 3.12. Model showing the relationship between improper fractions and mixed numbers

In grade 4, students progress in their understanding of the relationship between equivalent forms of the same fraction, such as $\frac{14}{3}$ and $4\frac{2}{3}$. As illustrated in figure 3.13, students can use the model of $4\frac{2}{3}$ to write an equivalent improper fraction. Students can see that 1 whole equals 3/3, so for every whole in the model, they need 3 thirds. So to write $4\frac{2}{3}$ as an improper fraction, they can join 4 groups of 3 thirds (or $4 \times 3 = 12$ thirds) and add the additional 2 thirds to make 14 thirds in all. Through these concrete and pictorial experiences, students begin to generalize that to change a mixed number to an improper fraction, they multiply the number of parts in the whole (or the denominator of the fraction part of the mixed number) by the number of wholes (or the whole-number part of the mixed number) and add the number of parts represented by the numerator in the mixed number. The result is the number of parts in the improper fraction (or the numerator of the improper fraction). The number of parts in the whole does not change, so the denominator of the improper fraction is the same.