

# Foreword

Teaching mathematics in prekindergarten–grade 12 requires a special understanding of mathematics. Effective teachers of mathematics think about and beyond the content that they teach, seeking explanations and making connections to other topics, both inside and outside mathematics. Students meet curriculum and achievement expectations when they work with teachers who know what mathematics is important for each topic that they teach.

The National Council of Teachers of Mathematics (NCTM) presents the Essential Understanding Series in tandem with a call to focus the school mathematics curriculum in the spirit of *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*, published in 2006, and *Focus in High School Mathematics: Reasoning and Sense Making*, released in 2009. The Essential Understanding books are a resource for individual teachers and groups of colleagues interested in engaging in mathematical thinking to enrich and extend their own knowledge of particular mathematics topics in ways that benefit their work with students. The topic of each book is an area of mathematics that is difficult for students to learn, challenging to teach, and critical for students' success as learners and in their future lives and careers.

Drawing on their experiences as teachers, researchers, and mathematicians, the authors have identified the big ideas that are at the heart of each book's topic. A set of essential understandings—mathematical points that capture the essence of the topic—fleshes out each big idea. Taken collectively, the big ideas and essential understandings give a view of a mathematics that is focused, connected, and useful to teachers. Links to topics that students encounter earlier and later in school mathematics and to instruction and assessment practices illustrate the relevance and importance of a teacher's essential understanding of mathematics.

On behalf of the Board of Directors, I offer sincere thanks and appreciation to everyone who has helped to make this series possible. I extend special thanks to Rose Mary Zbiek for her leadership as series editor. I join the Essential Understanding project team in welcoming you to these books and in wishing you many years of continued enjoyment of learning and teaching mathematics.

Henry Kepner  
President, 2008–2010  
National Council of Teachers of Mathematics



# Preface

From prekindergarten through grade 12, the school mathematics curriculum includes important topics that are pivotal in students' development. Students who understand these ideas cross smoothly into new mathematical terrain and continue moving forward with assurance.

However, many of these topics have traditionally been challenging to teach as well as learn, and they often prove to be barriers rather than gateways to students' progress. Students who fail to get a solid grounding in them frequently lose momentum and struggle in subsequent work in mathematics and related disciplines.

The Essential Understanding Series identifies such topics at all levels. Teachers who engage students in these topics play critical roles in students' mathematical achievement. Each volume in the series invites teachers who aim to be not just proficient but outstanding in the classroom—teachers like you—to enrich their understanding of one or more of these topics to ensure students' continued development in mathematics.

## **How much do you need to know?**

To teach these challenging topics effectively, you must draw on a mathematical understanding that is both broad and deep. The challenge is to know considerably more about the topic than you expect your students to know and learn.

Why does your knowledge need to be so extensive? Why must it go above and beyond what you need to teach and your students need to learn? The answer to this question has many parts.

To plan successful learning experiences, you need to understand different models and representations and, in some cases, emerging technologies as you evaluate curriculum materials and create lessons. As you choose and implement learning tasks, you need to know what to emphasize and why those ideas are mathematically important.

While engaging your students in lessons, you must anticipate their perplexities, help them avoid known pitfalls, and recognize and dispel misconceptions. You need to capitalize on unexpected classroom opportunities to make connections among mathematical ideas. If assessment shows that students have not understood the material adequately, you need to know how to address weaknesses that you have identified in their understanding. Your understanding must be sufficiently versatile to allow you to represent the mathematics in different ways to students who don't understand it the first time. In addition, you need to know where the topic fits in

the full span of the mathematics curriculum. You must understand where your students are coming from in their thinking and where they are heading mathematically in the months and years to come.

Accomplishing these tasks in mathematically sound ways is a tall order. A rich understanding of the mathematics supports the varied work of teaching as you guide your students and keep their learning on track.

### How can the Essential Understanding Series help?

The Essential Understanding books offer you an opportunity to delve into the mathematics that you teach and reinforce your content knowledge. They do not include materials for you to use directly with your students, nor do they discuss classroom management, teaching styles, or assessment techniques. Instead, these books focus squarely on issues of mathematical content—the ideas and understanding that you must bring to your preparation, in-class instruction, one-on-one interactions with students, and assessment.

### How do the authors approach the topics?

For each topic, the authors identify “big ideas” and “essential understandings.” The big ideas are mathematical statements of overarching concepts that are central to a mathematical topic and link numerous smaller mathematical ideas into coherent wholes. The books call the smaller, more concrete ideas that are associated with each big idea *essential understandings*. They capture aspects of the corresponding big idea and provide evidence of its richness.

The big ideas have tremendous value in mathematics. You can gain an appreciation of the power and worth of these densely packed statements through persistent work with the interrelated essential understandings. Grasping these multiple smaller concepts and through them gaining access to the big ideas can greatly increase your intellectual assets and classroom possibilities.

In your work with mathematical ideas in your role as a teacher, you have probably observed that the essential understandings are often at the heart of the understanding that you need for presenting one of these challenging topics to students. Knowing these ideas very well is critical because they are the mathematical pieces that connect to form each big idea.

### How are the books organized?

Every book in the Essential Understanding Series has the same structure:

- The introduction gives an overview, explaining the reasons for the selection of the particular topic and highlighting some of the differences between what teachers and students need to know about it.
- Chapter 1 is the heart of the book, identifying and examining the big ideas and related essential understandings.

Big ideas and essential understandings are identified by icons in the books.



marks a big idea,  
and



marks an essential  
understanding.

- Chapter 2 reconsiders the ideas discussed in chapter 1 in light of their connections with mathematical ideas within the grade band and with other mathematics that the students have encountered earlier or will encounter later in their study of mathematics.
- Chapter 3 wraps up the discussion by considering the challenges that students often face in grasping the necessary concepts related to the topic under discussion. It analyzes the development of their thinking and offers guidance for presenting ideas to them and assessing their understanding.

The discussion of big ideas and essential understandings in chapter 1 is interspersed with questions labeled “Reflect.” It is important to pause in your reading to think about these on your own or discuss them with your colleagues. By engaging with the material in this way, you can make the experience of reading the book participatory, interactive, and dynamic.

Reflect questions can also serve as topics of conversation among local groups of teachers or teachers connected electronically in school districts or even between states. Thus, the Reflect items can extend the possibilities for using the books as tools for formal or informal experiences for in-service and preservice teachers, individually or in groups, in or beyond college or university classes.

### A new perspective

The Essential Understanding Series thus is intended to support you in gaining a deep and broad understanding of mathematics that can benefit your students in many ways. Considering connections between the mathematics under discussion and other mathematics that students encounter earlier and later in the curriculum gives the books unusual depth as well as insight into vertical articulation in school mathematics.

The series appears against the backdrop of *Principles and Standards for School Mathematics* (NCTM 2000), *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (NCTM 2006), *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009), and the Navigations Series (NCTM 2001–2009). The new books play an important role, supporting the work of these publications by offering content-based professional development.

The other publications, in turn, can flesh out and enrich the new books. After reading this book, for example, you might select hands-on, Standards-based activities from the Navigations books for your students to use to gain insights into the topics that the Essential Understanding books discuss. If you are teaching students in prekindergarten through grade 8, you might apply your deeper understanding as you present material related to the three focal

points that *Curriculum Focal Points* identifies for instruction at your students' level. Or if you are teaching students in grades 9–12, you might use your understanding to enrich the ways in which you can engage students in mathematical reasoning and sense making as presented in *Focus in High School Mathematics: Reasoning and Sense Making*.

An enriched understanding can give you a fresh perspective and infuse new energy into your teaching. We hope that the understanding that you acquire from reading the book will support your efforts as you help your students grasp the ideas that will ensure their mathematical success.

The authors of the present volume wish to thank all of their reviewers, whose insightful comments caused them to think more deeply about important mathematics. They are especially grateful for the contributions of Richard Askey, Joanne Rossi Becker, Sandy Blount, Helen M. Doerr, Gail Keith, and Louis Lim.

# Introduction

This book focuses on ideas about functions. These are ideas that you need to understand thoroughly and be able to use flexibly to be highly effective in your teaching of mathematics in grades 9–12. The book discusses many mathematical ideas that are common in high school curricula, and it assumes that you have had a variety of mathematics experiences that have motivated you to delve into—and move beyond—the mathematics that you expect your students to learn.

The book is designed to engage you with these ideas, helping you to develop an understanding that will guide you in planning and implementing lessons and assessing your students' learning in ways that reflect the full complexity of functions. A deep, rich understanding of these relationships will enable you to communicate their influence and scope to your students, showing them how these ideas permeate the mathematics that they have encountered—and will continue to encounter—throughout their school mathematics experiences.

The understanding of functions that you gain from this focused study thus supports the vision of *Principles and Standards for School Mathematics* (NCTM 2000): “Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction” (p. 3). This vision depends on classroom teachers who “are continually growing as professionals” (p. 3) and routinely engage their students in meaningful experiences that help them learn mathematics with understanding.

## Why Functions?

Like the topics of all the volumes in NCTM's Essential Understanding Series, functions compose a major area of school mathematics that is crucial for students to learn but challenging for teachers to teach. Students in grades 9–12 need to understand functions well if they are to succeed in courses that build on quantitative thinking and relationships. Learners often have a narrow view of functions. On the basis of their frequent use of linear and quadratic functions, students tend to limit the concept of functions to equations or orderly rules. They frequently overlook many-to-one correspondences or irregular functions that could be very useful in describing or representing real-world phenomena. The importance of understanding functions and the challenge of understanding them well make them essential for teachers of mathematics in grades 9–12 to understand extremely well themselves.

Your work as a high school teacher of mathematics calls for a solid understanding of the mathematics that you—and your school, your district, and your state curriculum—expect your students to learn about functions. Your work also requires you to know how this mathematics relates to other mathematical ideas that your students will encounter in the lesson at hand, the current school year, and beyond. Rich mathematical understanding guides teachers' decisions in much of their work, such as choosing tasks for a lesson, posing questions, selecting materials, ordering topics and ideas over time, assessing the quality of students' work, and devising ways to challenge and support their thinking.

## Understanding Functions

Teachers teach mathematics because they want others to understand it in ways that will contribute to success and satisfaction in school, work, and life. Helping your high school students develop a robust and lasting understanding of functions requires that you understand this mathematics deeply. But what does this mean?

It is easy to think that understanding an area of mathematics, such as functions, means knowing certain facts, being able to solve particular types of problems, and mastering relevant vocabulary. For example, as a teacher at the secondary level, you are expected to know facts such as “functions are single-valued mappings from one set to another” and “linear functions have a constant rate of change.” It is likely that you need to be able to solve problems that involve tasks such as graphing a function or evaluating a function at a specific value. Your mathematical vocabulary is assumed to include such terms as *slope*, *maximum*, *minimum*, and *continuity*.

Obviously, facts, vocabulary, and techniques for solving certain types of problems are not all that you are expected to know about functions. In your ongoing work with students, you have undoubtedly discovered that you need to distinguish among different types of problems and know when particular strategies apply. For example, you must know the difference between linear and quadratic functions and the advantages and limitations of algebraic, tabular, and graphic representations of a function. You are also expected to be able to sort functions into families with common properties and describe the rate of change for particular functions.

It is also easy to focus on a very long list of mathematical ideas that all teachers of mathematics in grades 9–12 are expected to know and teach about functions. Curriculum developers often devise and publish such lists. However important the individual items might be, these lists cannot capture the essence of a rich understanding of the topic. Understanding this area deeply requires you not only to know important mathematical ideas but also to

recognize how these ideas relate to one another. Your understanding continues to grow with experience and as a result of opportunities to embrace new ideas and find new connections among familiar ones.

Furthermore, your understanding of functions should transcend the content intended for your students. Some of the differences between what you need to know and what you expect them to learn are easy to point out. For instance, you need to understand rate of change in a way that connects with proportional reasoning in middle school as well as more advanced problems that analyze the rate of change in differential and integral calculus at the college level.

Other differences between the understanding that you need to have and the understanding that you expect your students to acquire are less obvious, but your experiences in the classroom have undoubtedly made you aware of them at some level. For example, how many times have you been grateful to have an understanding of functions that enables you to recognize the merit in a student's unanticipated mathematical question or claim? How many other times have you wondered whether you could be missing such an opportunity or failing to use it to full advantage because of a gap in your knowledge?

As you have almost certainly discovered, knowing and being able to do familiar mathematics are not enough when you're in the classroom. You also need to be able to identify and justify or refute novel claims. These claims and justifications might draw on ideas or techniques that are beyond the mathematical experiences of your students and current curricular expectations for them. Consider, for example, a claim that the popular technique of using a vertical line test to determine functionality is not applicable to all functions or the idea that sequences are functions.

## Big Ideas and Essential Understandings

Thinking about the many particular ideas that are part of a rich understanding of functions can be an overwhelming task. Articulating all of those mathematical ideas and their connections would require many books. To choose which ideas to include in this book, the authors considered a critical question: What is *essential* for teachers of mathematics in grades 9–12 to know about functions to be effective in the classroom? To answer this question, the authors drew on a variety of resources, including personal experiences, the expertise of colleagues in mathematics and mathematics education, and the reactions of reviewers and professional development providers, as well as ideas from curricular materials and research on mathematics learning and teaching.

As a result, the mathematical content of this book focuses on essential knowledge for teachers about functions. In particular, chapter 1 is organized by five big ideas related to this important area of mathematics. Each of these big ideas is supported by smaller, more specific mathematical ideas, which the book calls *essential understandings*. This book focuses on three to seven interconnected essential understandings that are related to each big idea. These ideas elaborate what you need to know for an understanding of functions. Gaining this understanding is an extremely valuable and useful accomplishment because functions help us to think quantitatively about real-world phenomena. They provide us with ways to think about relationships—particularly change within those relationships.

## Benefits for Teaching, Learning, and Assessing

An understanding of functions can help you implement the Teaching Principle enunciated in *Principles and Standards for School Mathematics*. This Principle sets a high standard for instruction: “Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well” (NCTM 2000, p. 16). As in teaching about other critical topics in mathematics, teaching about functions requires knowledge that goes “beyond what most teachers experience in standard preservice mathematics courses” (p. 17).

Chapter 1 comes into play at this point, offering an overview of functions that is intended to be more focused and comprehensive than many discussions of the topic that you are likely to have encountered. This chapter enumerates, expands on, and gives examples of the big ideas and essential understandings related to functions, with the goal of supplementing or reinforcing your understanding. Thus, chapter 1 aims to prepare you to implement the Teaching Principle fully as you provide the support and challenge that your students need for robust learning about functions.

Consolidating your understanding in this way also prepares you to implement the Learning Principle outlined in *Principles and Standards*: “Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge” (NCTM 2000, p. 20). To support your efforts to help your students learn about functions in this way, chapter 2 builds on the understanding of functions that chapter 1 communicates by pointing out specific ways in which the big ideas and essential understandings connect with mathematics that students typically encounter earlier or later in school. This chapter supports the Learning Principle by emphasizing longitudinal connections in students’ learning about functions.

For example, foundations for ideas of function begin to develop in elementary school, where students study patterns and develop an informal notion of variable. By middle school, they are learning to represent patterns in tables and to look at covariation. In high school, students analyze these relationships and construct families of functions by studying characteristics of specific functions. In college, they extend their acquaintance with the domains of functions to include complex numbers, which may expand their notions about family characteristics.

The understanding that chapters 1 and 2 convey can strengthen another critical area of teaching. Chapter 3 addresses this area, building on the first two chapters to show how an understanding of functions can help you select and develop appropriate tasks, techniques, and tools for assessing your students' understanding of the topic. An ownership of the big ideas and essential understandings related to functions, reinforced by an awareness of students' past and future experiences with the ideas, can help you ensure that assessment in your classroom supports the learning of significant mathematics.

Such assessment satisfies the first requirement of the Assessment Principle set out in *Principles and Standards* (NCTM 2000): "Assessment should support the learning of important mathematics and furnish useful information to both teachers and students" (p. 22). An understanding of functions can also help you satisfy the second requirement of the Assessment Principle, by enabling you to develop assessment tasks that give you specific information about what your students are thinking and what they understand. For instance, you could ask a student to create (or select) equations that represent a given set of related graphs. This reversal of the common "graph this equation" assessment item can help you assess several of the essential understandings related to multiple big ideas of functions with one item.

## Ready to Begin

This introduction has painted the background, preparing you for the big ideas and associated essential understandings related to functions that you will encounter and explore in chapter 1. Reading the chapters in the order in which they appear can be a very useful way to approach the book. Read chapter 1 in more than one sitting, allowing time for reflection. Absorb the ideas—both big ideas and essential understandings—that contribute to an understanding of functions. Appreciate the connections among these ideas. Carry your newfound or reinforced understanding to chapter 2, which guides you in seeing how the ideas related to functions are connected to the mathematics that your students have encountered

earlier or will encounter later in school. Then read about teaching, learning, and assessment issues in chapter 3.

Alternatively, you may want to take a look at chapter 3 before engaging with the mathematical ideas in chapters 1 and 2. Having the challenges of teaching, learning, and assessment issues clearly in mind, along with possible approaches to them, can give you a different perspective on the material in the earlier chapters.

No matter how you read the book, let it serve as a tool to expand your understanding, application, and enjoyment of functions.