Navigations Series

Pre-K–Kindergarten

Problem Solving and Reasoning

Introduction

In three landmark publications—*Agenda for Action* (NCTM 1980), *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989), and *Principles and Standards for School Mathematics* (NCTM 2000)—the National Council of Teachers of Mathematics has consistently identified learning to solve problems as the major goal of school mathematics. Each of these publications highlights the importance of giving students opportunities to apply the mathematical concepts and skills that they are learning—together with various problem-solving strategies and methods of reasoning—to the solution of challenging problems. The hope is that students will gain a greater appreciation for the power of mathematics and for their abilities to wrestle with important mathematical ideas. Neither the mathematical knowledge nor the reasoning strategies can be developed in isolation. They must be learned and used concurrently. Furthermore, problem-solving strategies and reasoning methods are rarely applied in isolation from each other; they, too, are normally applied together in solving mathematical problems.

Problem-Solving Strategies and Reasoning Methods

Students begin to develop a variety of problem-solving strategies and reasoning methods in prekindergarten through grade 2. These strategies and methods are illustrated here with examples from the investigations in this book.

Problem-Solving Strategies and Reasoning Methods

- Identification of relationships
- Inference
- Generalization
- Representation
- Guess, check, and revise
- Analogy
- Verification

Identification of mathematical relationships

Determining how numbers, shapes, and mathematical concepts are related is central to understanding mathematics. Early in the learning of mathematics, students identify the characteristics of shapes in order to make comparisons. They look for similarities and differences among objects and numbers, and they sort, categorize, rank, or sequence them on the basis of attributes. Later, students differentiate among problems by noting their structural similarities and differences. At the most abstract level, students identify mathematical relationships presented symbolically or in tables, graphs, diagrams, models, or text.

Students gain experience in identifying mathematical relationships in the investigations Bears in the House and in the Park, Shape Families, and Glyph Gallery. Students model story problems in Bears in the House and in the Park and decide whether to add, subtract, or use skip counting or repeated addition to solve the problems. The students base their decisions about which operations to use on their analysis of the problems and on the relationship between the data and the questions. In Shape Families, students identify characteristics of shapes and group shapes by like features. Students identify and compare some physical characteristics of people in Glyphs Gallery, and then they determine the relationships between the characteristics and other attributes, like gender, age, number of hair strands, and preference for pizza.

Inference

Inference is the strategy of deducing unstated information from observed or stated information. Students use inferential reasoning when they formulate conjectures or hypotheses or draw conclusions from their analyses of a problem.

Prekindergarten and kindergarten students are introduced to inferential reasoning in the investigations Fire Trucks and Hats, Shape Families, Line Up, and Glyph Gallery. As parts of a pattern are revealed in Fire Trucks and Hats, students make inferences about the identity of the unseen elements in the pattern. They then determine if their inferences are correct when those elements are revealed. In Shape Families, students analyze shapes and make inferences about how some shapes are alike and others are different. In Line Up, students solve logic problems involving the comparison of measurements. By analyzing glyphs in Glyph Gallery, they make inferences and draw conclusions about relationships between facial features and age, gender, house color, and pizza preference.

Generalization

Generalization is the strategy of identifying a pattern of information or events and then using the pattern to formulate conclusions about other like situations. Students generalize when they—

- identify and continue shape, number, rhythm, color, and pitch patterns;
- describe these patterns with rules in words or symbols;

- predict from a sample; and
- identify trends from sets of data.

In Fire Trucks and Hats, students identify the relationships among the elements in a repeating pattern of pictures, describe and continue the pattern, and then generalize the pattern by representing it with different types of elements.

Representation

Representation is the process of using symbols, words, illustrations, graphs, and charts to characterize mathematical concepts and ideas. It involves creating, interpreting, and linking various forms of information and data displays, including those that are graphic, textual, symbolic, three-dimensional, sketched, or simulated. The process also involves identifying the most appropriate display for a particular situation, purpose, and audience, and it requires the ability to translate among different representations of the same relationship.

Students develop an understanding of representation in the investigations Bears in the House and in the Park, Fire Trucks and Hats, and Glyph Gallery. In Bears in the House and in the Park, students use chips to represent bears in modeling and solving story problems. They also make drawings to show solutions to problems. In Fire Trucks and Hats, students consider various ways in which patterns can be represented, and they identify patterns that are shown pictorially (e.g., hat-hat-truck-hat-hat-truck) and with letters (AABAAB) as the "same." Students interpret diagrams called *glyphs* in Glyph Gallery, and they create legends for existing glyphs.

Guess, check, and revise

This strategy involves using one or more conditions of a problem to identify a candidate for the solution to the problem, checking the candidate against all the problem conditions, and revising the candidate appropriately if it does not meet all the conditions. The revised candidate for the solution is then checked against the problem conditions. The process continues until a solution that matches all the problem conditions is found.

In Line Up, students are presented with three people and a set of clues about the order of the people in a line. The students line up the people according to a possible order that meets one of the problem conditions, and then they compare the lineup with the clues. If the lineup does not correspond with all the clues, the students revise the order and check it again.

Analogy

Analogy is a method of identifying structural similarities and important elements in problems without regard to the particular contexts. Analogy facilitates the solution process because known or easily identified solutions to a simpler problem can be applied to a more complex problem. For instance, if students recognize that two problems are structurally alike and they know how to solve one of the problems, they can apply the same solution method to the other problem. In another example, when students are confronted with a complex mathematical problem, they may construct a simpler problem that preserves the essential features or properties of the more difficult problem. By solving the simpler problem first, the students may discover a solution method that can be applied to the more complex problem.

In Bears in the House and in the Park, students are encouraged to identify problems that have the same mathematical structure and to apply the same solution strategy to those problems. This investigation sets the stage for developing students' analogical reasoning.

Verification

Verification is the process of checking, proving, or confirming a conclusion or point of view. Verification occurs when students—

- identify information that is relevant to, and has value for, the solution of a problem (and when they disregard irrelevant information);
- identify fallacies and unwarranted assumptions;
- recognize that solutions are reasonably close to estimates and make sense within the contexts of problems;
- justify the use of particular solution strategies by convincing arguments or—at a later age—proofs;
- formulate counterexamples.

Students also verify their own solutions when they identify gaps, inconsistencies, or contradictions in another person's line of reasoning.

Verification is developed in all the investigations in this book. In Bears in the House and in the Park, students create story problems, solve one another's problems, and verify their partner's solution. When they predict the identity of covered elements in patterns in Fire Trucks and Hats, students must justify their solutions. In Shape Families, students justify the grouping of similar shapes and the elimination of the shape that is different. Students verify the order of people in Line Up by checking the order against the problem conditions. In Glyph Gallery, after determining the relationship between glyphs and attributes of objects and creating a glyph legend, students check to be sure that all the glyphs are interpreted correctly.

Developing Mathematical Dispositions

It is hoped that these investigations, which emphasize problem solving and reasoning, and other challenging mathematical activities will develop students' love of mathematics and their dispositions to—

- enjoy solving difficult problems;
- make sense of seemingly nonsensical situations or fix or "salvage" vague problems by rephrasing them and eliminating ambiguities;
- persist until they find a solution to a problem or until they determine that no solution exists;

- reflect on their solutions and solution methods and make adjustments accordingly;
- recognize that to solve some problems, they must learn more mathematics;
- generate new mathematical questions for a given problem;
- listen to others and analyze and verify their peers' lines of reasoning.

The Role of the Teacher

To strengthen students' mathematical reasoning and problem-solving abilities, teachers must create classroom environments that are mathematically "safe"—that is, ones in which every child feels free to make conjectures, to explore different ways of thinking, and to share his or her ideas with classmates. Teachers must be able to assess students' thinking and adjust mathematical tasks on the basis of assessment data. Most important, teachers must facilitate classroom discourse and ask probing questions in order to deepen students' understanding of the mathematics and of the reasoning methods and problem-solving strategies that the students employ.

Facilitate classroom discourse

Classroom discourse gives students opportunities to communicate their mathematical reasoning. In such discourse, students explore conjectures and clarify their understanding of problem-solving strategies. Informal discussions among pairs or small groups of students can enhance students' commitments to a task and assist less able learners in understanding the nature of a task, the meaning of the terminology, and the appropriate vocabulary to use in a response. Whole-class discussions serve as forums for students to share their findings, make generalizations, and explore alternative approaches. Classroom discourse also gives teachers important insights into their students' thinking.

Students in prekindergarten to grade 2 often share their mathematical thinking in pairs or small groups quite naturally, with little or no intervention by the teacher. Most young children are comfortable talking aloud as they solve problems. It can be challenging, however, to sustain a whole-class discussion among young students. Nonetheless, teachers can foster such discussions in a variety of ways:

- *Extend wait time.* Students need time to ponder important ideas and to formulate their responses. Don't be concerned if your students don't comment immediately. When teachers wait a bit longer than they are accustomed to doing, students often do respond.
- *Allow students to correct one another.* It can be difficult not to respond to every incorrect comment. Constant correction by the teacher, however, leads students to rely on the teacher as the authority rather than on their own mathematical knowledge, reasoning, and verification methods.
- *Ask more questions.* Instead of always responding to a student's contribution with a direct comment, encourage student-to-student interaction by asking such questions as these: "Did anyone else find

this solution?" "Can anyone help with this question?" "What do you think we should do about this?"

- *Support reticent speakers.* Afford students who rarely comment or ask questions opportunities to practice what they intend to share with their group or class so that they may become more confident. Inquire if they would like to speak first so that they don't need to wait anxiously for their turns. You can also bring these students into discussions by asking, "Would anyone else like to add something or give another opinion?"
- *Encourage the use of recording sheets.* For very young children, recording may take the form of making simple drawings to record solutions, strategies, or merely something about the problem. As students' abilities to record their thinking develop, drawings become more sophisticated, and recordings may include written explanations and symbolic representations. More-mature students may depict more than one solution strategy. The recording sheets give all students something to share and can help young children recall their investigative work.
- *Summarize ideas.* Recording students' ideas on the chalkboard or on large easel paper helps focus discussions and lets the students know that their ideas are important.

Students' discourse is an invaluable resource. It can lead to a deeper understanding of the mathematics embedded in problems and may launch new investigations. It offers opportunities for students to develop their reasoning abilities as they challenge and defend ideas. Finally, it gives teachers insights into students' thinking that can in turn be valuable in making instructional decisions.

Ask probing questions

The questions that a teacher asks during an investigation can help students understand their own thinking. In responding to these questions, the students make links among problems, strategies, and representations, and they check their logic and make generalizations.

"Good problem solvers monitor their thinking regularly and automatically."
(Van de Walle 2004, p. 54)

Good problem solvers know what they are doing and why they are doing it. They know when they need help or should change strategies. Teachers' questions help young students develop good metacognitive habits. The following are examples of questions that prompt students' reflection:

- "What did you do first? Why?"
- "Why did you change your mind?"
- "What were you thinking when you recorded this?"
- "Which clue did you think was the most (least) helpful? Why?"
- "What made this investigation easy (or difficult) for you?"
- "What do you plan to do next?"
- "What hint would you give to a friend who was stuck?"

Discovering connections among problems, strategies, and representations deepens mathematical thinking and strengthens problem-solving abilities. To help students make such connections, ask questions such as these:

- "Does this problem remind you of another problem that you have already solved?"
- "Is there another way to solve this problem?"
- "Can you create a problem that could also be solved this way?"
- "Can you represent this information in a different way?"

Rich mathematical investigations give students opportunities to develop their reasoning skills further. Students can make predictions, generalize ideas, and recognize logical inconsistencies. Questions such as the following can help students enhance their reasoning abilities:

- "What do you think will happen next? Why?"
- "Do you think this pattern will continue? Why?"
- "Would this still be true if you began with an odd number [*or other counterexample*]?"
- "Can you state a general rule you have discovered?"
- "What will never happen when you do this?"

Finally, through your example, you can strengthen your students' problem-solving and reasoning abilities. Throughout the school day, teachers as well as students have numerous opportunities to exhibit curiosity about how things work and what generalizations can be made, to exemplify good reasoning and the use of varying problem-solving strategies, and to evoke in students the belief that mathematical thinking is an elegant and exciting problem-solving tool.