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Mathematical Reasoning and Sense Making[1]

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Reasoning and sense making are the foundation of mathematical competence and proficiency, and their absence leads to failure and disengagement in mathematics instruction. Thus, developing students' capabilities with reasoning and sense making should be the primary goal of mathematics instruction. In order to achieve this goal, all mathematics classes should provide ongoing opportunities for students to implement these processes.

What are mathematical reasoning and sense making? *Reasoning* is the process of manipulating and analyzing objects, representations, diagrams, symbols, or statements to draw conclusions based on evidence or assumptions. *Sense making* is the process of understanding ideas and concepts in order to correctly identify, describe, explain, and apply them. Genuine sense making makes mathematical ideas "feel" clear, logical, valid, or obvious. The moment of sense making is often signaled by exclamations such as "Aha!" "I get it!" or "Oh, I see!"

Why Focus on Reasoning and Sense Making?

Reasoning and sense making are critical in mathematics learning because students who genuinely make sense of mathematical ideas can apply them in problem solving and unfamiliar situations and can use them as a foundation for future learning. Even with mathematical skills, "[i]n order to learn skills so that they are remembered, can be applied when they are needed, and can be adjusted to solve new problems, they must be learned with understanding [i.e., they must make sense]" (Hiebert et al. 1997, p. 6).

Sense making is also important because it is an intellectually satisfying experience, and not making sense is frustrating (Hiebert et al. 1997). Students who achieve genuine understanding and sense making of mathematics are likely to stay engaged in learning it. Students who fail to understand and make sense of mathematical ideas and instead resort to rote learning will eventually experience continued failure and withdraw from mathematics learning.

Understanding How Students Think

An abundance of research describing how students learn mathematics indicates that effective mathematics instruction is based on the following three principles (Battista 2001; Bransford, Brown, and Cocking 1999; De Corte, Greer, and Verschaffel 1996; Greeno, Collins, and Resnick 1996; Hiebert and Carpenter 1992; Lester 1994; NRC 1989; Prawat 1999; Romberg 1992 Schoenfeld 1994; Steffe and Kieren 1994):

1. To genuinely understand mathematical ideas, students must construct these ideas for themselves as they intentionally try to make sense of situations; their success in constructing the meaning of new mathematical ideas is determined by their preexisting knowledge and types of reasoning and by their commitment to making personal sense of those ideas.
2. To be effective, mathematics teaching must carefully guide and support students as they attempt to construct personally meaningful mathematical ideas in the context of problem solving, inquiry, and student discussion of multiple problem-solving strategies. This sense-making and discussion approach to teaching can increase equitable student access to powerful mathematical ideas, as long as it regularly uses embedded formative assessment to determine the amount of guidance each student needs. (Some students construct ideas quite well with little guidance other than well-chosen sequences of problems; other students need more direct guidance, sometimes in the form of explicit description.)
3. To effectively guide and support students in constructing the meaning of mathematical ideas, instruction must be derived from research-based descriptions of how students develop reasoning about particular mathematical topics (such as those given in research-based learning progressions).

Consistent with this view on learning and teaching, professional recommendations and research suggest that mathematics teachers should possess extensive research-based knowledge of students' mathematical thinking (An, Kulm and Wu 2004; Carpenter and Fennema 1991; Clarke and Clarke 2004; Fennema and Franke 1992; Saxe et al. 2001; Schifter 1998; Tirosh 2000). Teachers should "be aware of learners' prior knowledge about particular topics and how that knowledge is organized and structured" (Borko and Putnam 1995, p. 42). And, because numerous researchers have found that students' development of understanding of particular mathematical ideas can be characterized in terms of developmental sequences or *learning progressions* (e.g., Battista and Clements 1996; Battista et al. 1998; Cobb and Wheatley 1988; Steffe 1992; van Hiele 1986), teachers must understand these learning progressions. They must understand "the general stages that students pass through in acquiring the concepts and procedures in the domain, the processes that are used to solve different problems at each stage, and the nature of the knowledge that underlies these processes" (Carpenter and Fennema 1991, p. 11). Research clearly shows that teacher use of such knowledge improves students' learning (Fennema and Franke 1992; Fennema et al. 1996). "There is a good deal of evidence that learning is enhanced when teachers pay attention to the knowledge and beliefs that learners bring to a learning task, use this knowledge as a starting point for new instruction, and monitor students' changing conceptions as instruction proceeds" (Bransford et al. 1999, p. 11).

Beyond understanding the development of students' mathematical reasoning, it is important to recognize that to be truly successful in learning mathematics, students must stay engaged in making personal sense of mathematical ideas. To stay engaged in mathematical sense making, students must be successful in solving *challenging but doable* problems. Such problems strike a delicate balance between involving students in the hard work of careful mathematical reasoning and having students succeed in problem solving, sense making, and learning. Keeping students successfully engaged in mathematical sense making requires us to understand each student's mathematical thinking well enough to continuously engage him or her in *successful* mathematical sense making. Furthermore, to pursue mathematical sense making during instruction, students must *believe*—based on their past experiences—that they are capable of making sense of mathematics. They must also believe that they are supposed to make sense of all the mathematical ideas discussed in their mathematics classes.

Finally, as part of the focus on reasoning and sense making in mathematics learning, students must adopt an inquiry disposition. Indeed, students learn more effectively when they adopt an active, questioning, inquiring frame of mind; such an inquiry disposition seems to be a natural characteristic of the mind's overall sense-making function (Ellis 1995; Feldman and Kalmar 1996).

Reaching All Students

The principled, student-reactive teaching described above not only helps all students maximize their learning but also benefits struggling students (Villasenor and Kepner 1993). In fact, this type of teaching supports all three tiers of Response to Intervention (RTI) instruction. For Tier 1 high-quality classroom instruction for all students, research-based instructional materials include extensive descriptions of the development of students' learning of particular mathematical topics. Research shows that teachers who understand such information about student learning teach in ways that produce greater student achievement. For Tier 2, research-based instruction enables teachers to better understand and monitor each student's mathematics learning through observation, embedded assessment, questioning, informal assessment during small-group work, and formative assessment. They can then choose instructional activities that meet their students' learning needs: whole-class tasks that benefit students at all levels or different tasks for small groups of students at the same level. For Tier 3, research-based assessments and learning progressions support student-specific instruction for struggling students so that they receive the long-term individualized instruction sequences they need.

Because extensive formative assessment is embedded in this type of teaching, support for its effectiveness also comes from research on the use of formative assessment, which indicates that formative assessment helps all students—and perhaps particularly struggling students—to produce significant learning gains, often reducing the learning gap between struggling students and their peers.

What Does Sense Making During Learning and Teaching Look Like?

The following sections explore two examples of the development of students' reasoning and sense making about particular mathematical ideas. We examine obstacles to sense making, variations in student sense making, and how teaching can support sense making at various levels of sophistication.

Making Sense of Place Value in Adding Two-Digit Numbers

To illustrate the nature of mathematical sense making and reasoning, consider how Bill, a second-grade student, approached the problem "What is 24 + 58?"

Bill: [*Writing out the traditional addition algorithm*] 8 + 4 is 12. Write the 2 here and the 1 up here. 1 + 2 + 4 = 7. Write the 7 next to the 2.

$$\begin{array}{r} \scriptstyle 1 \\ 24 \\ +48 \\ \hline 72 \end{array}$$

Teacher: Is this really a 1?

Bill: Yes, it came from the 12; you're not allowed to put it next to the 2.

Teacher: Why not?

Bill: You're just not allowed to do that.

Bill, like the majority of young students using this algorithm, showed no evidence of making conceptual sense of this procedure. His only justification was to cite some rule that a teacher or parent had given him, which he felt compelled to follow.

In contrast, there are many ways that students can make personal sense of the problem 24 + 58, as shown in the following class discussion. Over the school year in this second-grade class, students had made sense of two-digit addition in a variety of increasingly sophisticated ways.

Teacher: All year we have been talking about addition, and you have invented a bunch of ways to add two-digit numbers. Today, I want us to take a look at all these ways. I'd like you each to solve this problem in several different ways. Then, we'll talk about what you did.

Teacher: [*After students worked on the problem 24 + 48 by themselves at their seats*] Okay, I want to hear what kinds of ways you used to solve this problem, and I want you explain to the class why you did what you did.

Fred: I remember doing the problem like this: 20, 30, 40, 50, 60, 70, 72.

Teacher: Explain what you did.

Fred: I started with the 20 in 24 and counted tens in 48. Then I counted 10 more from the 12, then 2 more.

Teacher: Where did you get 12?

Fred: It's just 4 + 8 = 12. From the 24 and 48.

Teacher: Okay, how about a different way?

Mary: I did 48, 58, 68, 70, 72.

Teacher: Explain what you did.

Mary: I started with 48, and then I did the 24. I counted 10 onto 48, then 10 more, and then 2 + 2 makes 24. If you do one part at a time, it's easy.

Teacher: Okay, how about another way?

Jon: I did 40 + 20 = 60; 8 + 4 = 12; 60 + 12 = 72.

Teacher: How did you know you could do that?

Jon: You just add the tens, then add the ones, then add them together.

Teacher: Okay, how about another different way?

Serena: I need to write it. [*Goes to board and writes as shown below, then explains*] Here's how I added: 8 + 4 is 12. So I wrote a 2 from 12 under the 8 and put the 10 from 12 over the 20, because it's tens. Then I added 10 and 20 and 40, and I got 70. And 70 + 2 is 72.

$$\begin{array}{rl} & \mathit{10} \\ & 20 + 4 \\ + & \underline{40 + 8} \\ & 70 + 2 = 72 \end{array}$$

Teacher: Why did you write the 2 under the 8?

Serena: You just put the 2 ones from 12 in the ones place that's under the 4 and 8.

In this class discussion, we see that the students have made sense of adding 2 two-digit numbers in a wide variety of ways. In fact, the methods that these students used to solve this problem fit nicely into a *learning progression*, which is a description of the successively more sophisticated ways of reasoning and sense making that students pass through in developing a deep understanding of a mathematical idea.[2,3] Battista's learning progression for place value is outlined in chart 1.1. (See Battista 2012a for a much more detailed description, including sublevels, student examples, and suggestions for teaching students at each level.) The outline in chart 1.1 includes descriptions of the sense making and understanding of addition and subtraction algorithms that are possible for students at each learning progression level. Notice that it is not until students reach level 3 that they can even begin to truly make sense of algorithms for addition and subtraction of multi-digit numbers. Critically important is the fact that it is extremely rare for students to reach higher levels in the learning progression without first passing through the earlier levels.

No student in the previously described second-grade discussion exhibited level 5 reasoning (shown below), which we should not expect of many students until a later grade.

Jake: I wrote it like this [*writes below on the board*]. So, 8 + 4 = 12. That's 10, which I wrote up here, plus 2, which I wrote down here. Then 10 + 20 + 40 = 70, and I wrote the 7 in the tens place. So the answer is 72.

$$\begin{array}{r} \scriptstyle 10 \\ 24 \\ +48 \\ \hline 72 \end{array}$$

Teacher: Why did you write the 7 to the left of the 2?

Jake: I wrote 7 in the tens place because it's 70.

Premature Learning of Computational Algorithms

What happens to students' sense making when computational algorithms are taught before the students have progressed to an appropriate conceptual level in the learning progression? To see, we examine the reasoning of second grader Dion (dialogue excerpted from Battista 2012a).

Chart 1.1. Battista's (2012a) learning progression (LP) for place-value understanding of whole numbers

LP Level	Students' Conceptualization of Place Value	Possible Student Sense Making of Addition/ Subtraction Algorithms	Student Example
0	Student has difficulties counting by ones.	No sense making of place value or algorithms.	
1	Student operates on numbers as collections of ones (no skip-counting by place value).	Very little sense making of place value, and only rote use of algorithms possible.	Bill
2	Student operates on numbers by skip counting by place value (e.g., counts by tens).	Weak or no connection between place value and algorithms; only rote use of algorithms possible.	Fred Mary
3	Student operates on numbers by combining and separating place-value parts (e.g., adds tens parts without counting).	Explicit use of place value in informal multi-digit computation; emerging but incomplete understanding of place value in algorithms.	Jon
4	Student understands place value in expanded algorithms.	Place-value understanding of expanded algorithms (through hundreds).	Serena
5	Student understands place value in traditional algorithms.	Place-value understanding of traditional algorithms (through hundreds).	Jake

No student in the previously described second-grade discussion exhibited Level 5 reasoning (shown in the following dialogue), which we should not expect of many students until a later grade.

Teacher: What is the sum of 47 + 24?

Dion: [*Writing*] I added 7 and 4; it was 11. So then I put a 1 right here [*in solution*] and a 1 up here [*above 4 in 47*], and then I put 1 + 4 + 2 and it equals 7.

$$\begin{array}{r} \mathit{1} \\ 47 \\ \underline{+24} \\ 71 \end{array}$$

Teacher: Okay, and then you got 71?

Dion: Yeah.

Teacher: So what does this 1 [*above the 4 in 47*] stand for?

Dion: To add with the 4 and the 2.

Teacher: Is it just a 1?

Dion: Yes.

Teacher: Use the place-value blocks to solve the problem 36 + 28 [*shows the written problem*].

Dion: I've got 36 and 28. [*Picks up the 2 ten-blocks from 28*] 20. [*Picks up the 3 ten-rods from 36 and lays them down*] 30. [*Points to one of the 2 ten-blocks from 28 in his hand, then lays down the 2 ten-blocks*] 40, 50. [*Places the one-blocks from both piles together and counts them, one at a time*] 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64. [*Writes "64" on his paper*] 64.

Teacher: There are 25 squares under the card. How many squares are there altogether?

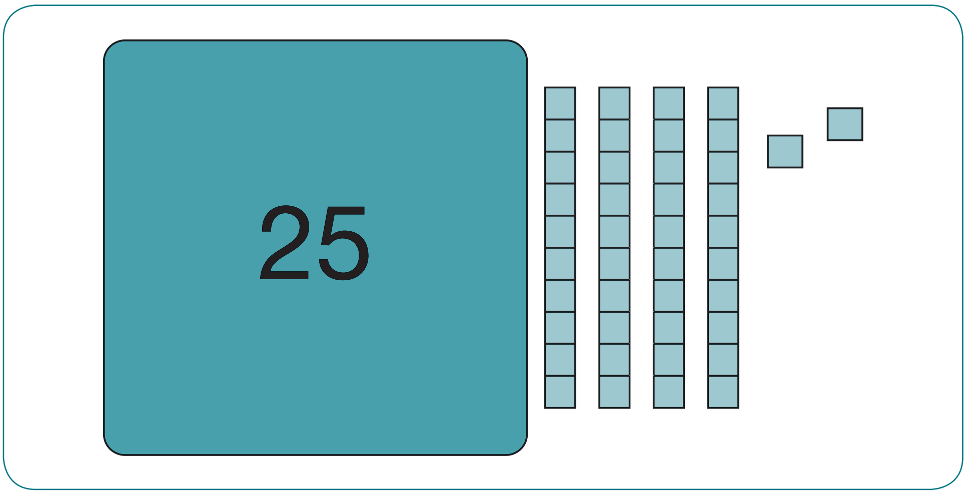

Fig. 1.1

Dion: So, 25 squares under this [*pointing at card*], and there'd be [*pointing at columns of ten and counting by tens, then counting the last 2 squares individually*] 35, 45, 55, 65, 66, 67. 67.

We see from these examples that Dion is reasoning at level 2 *when he has place-value blocks or pictorial support*. But when he does not have visual support, he uses the standard algorithm, which he has memorized by rote (using a type of level 1 reasoning). Also, Dion does not yet combine tens and ones, reasoning that is essential for making sense of an addition algorithm.

Instruction

To determine how to encourage and support Dion to increase the sophistication of his reasoning about place value in two-digit addition, we need to examine the learning progression in chart 1.1 (Battista 2012a). According to this learning progression, our first instructional step should be to help Dion use level 2 reasoning strictly mentally, without concrete or pictorial support; that is, we want to help Dion build on the current level 2 reasoning he uses with place-value blocks and diagrams. Our second instructional step should be to help Dion move to level 3 reasoning, once again, first with visual support and then without. At this point, we should not permit Dion to use a paper-and-pencil algorithm to solve these problems.

Step 1. Moving Dion to level 2 reasoning without visual support

Teacher: How many cubes are in each pile? Write the number under the piles.

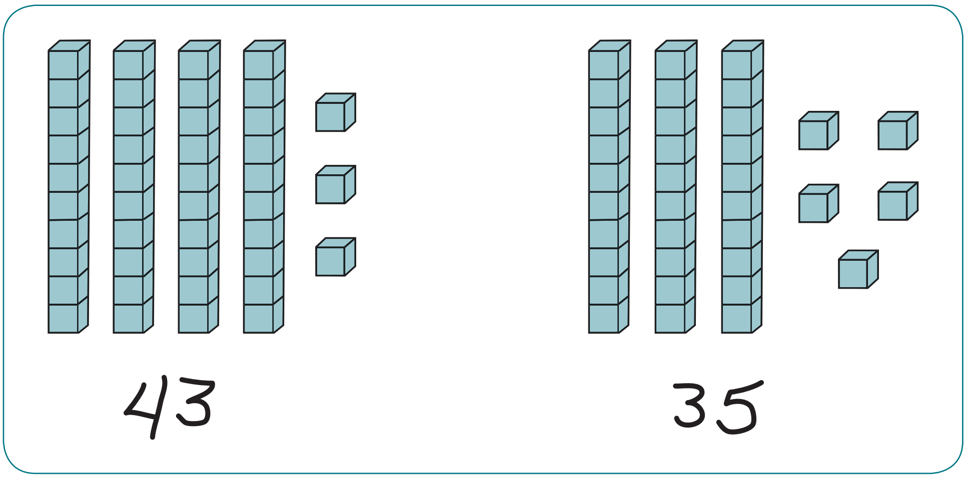

Fig. 1.2

Teacher: [*Covers the piles but not the numbers*] How many cubes are there altogether under the mats (fig. 1.3)? Can you count by tens and ones?

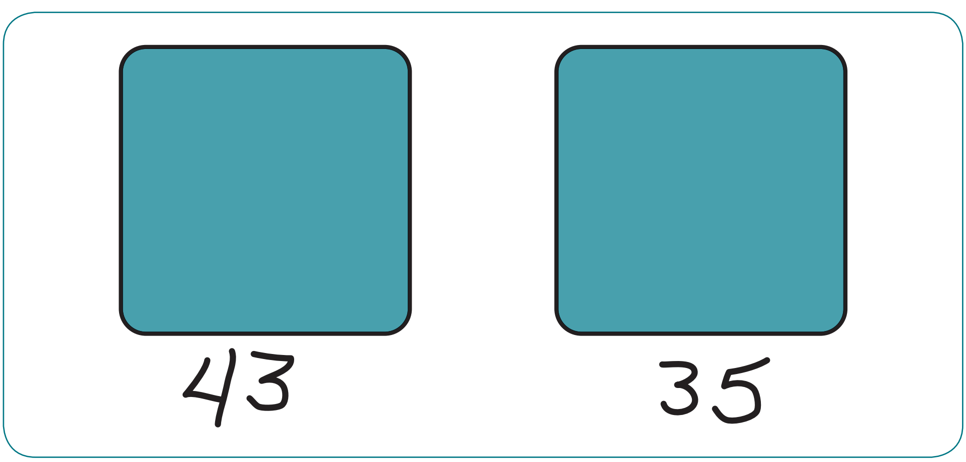

Fig. 1.3

If Dion cannot do the problem strictly mentally, without the help of visuals or pen and pencil, remove the right mat.

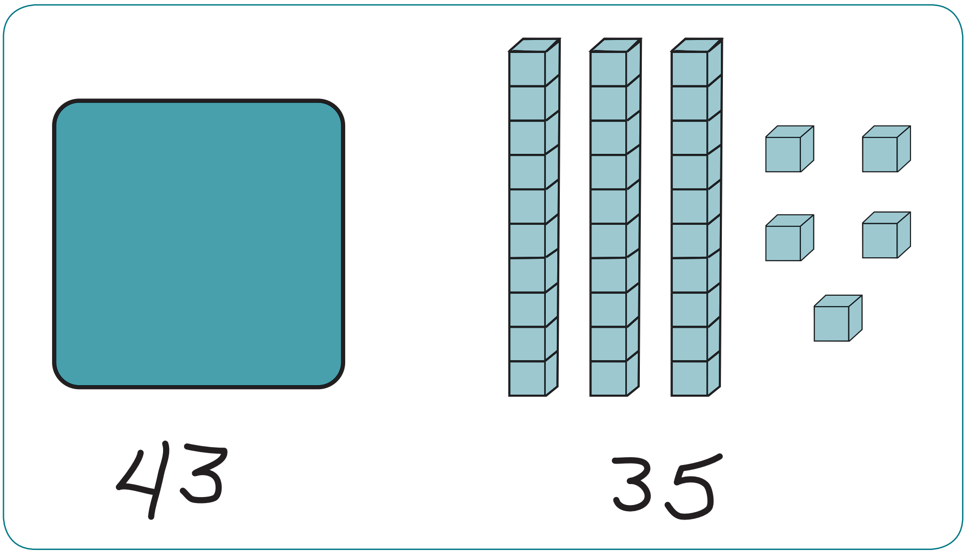

Fig. 1.4

Repeat this procedure until Dion can solve problems with both mats in place. Later present problems that are in written form but have visual support available if Dion needs it.

Step 2. Moving Dion to level 3 reasoning

Teacher: How many cubes are in each pile? Write the number under the piles. How many cubes are there altogether? Can you figure it out without counting by tens?

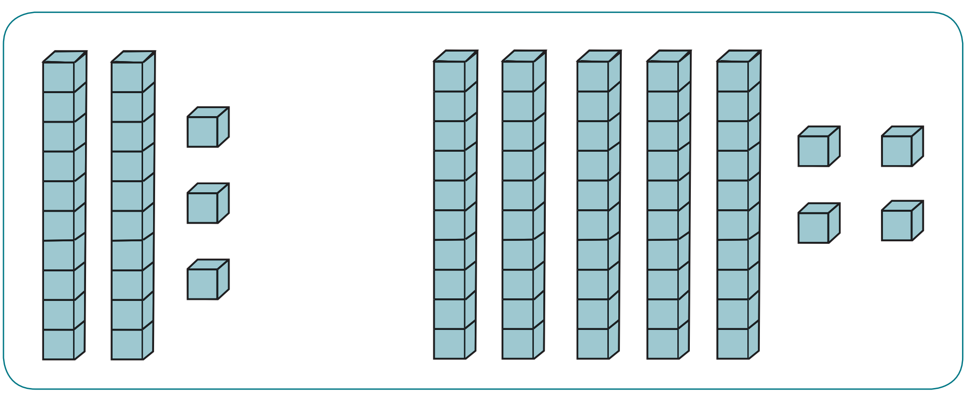

Fig. 1.5

If Dion cannot solve the problem without counting by tens, let him count by tens and then redo the problem (or do a different problem) while asking him questions that suggest and scaffold level 3 reasoning.

Teacher: How many ten-blocks are in 23 [*pointing to blocks for 23*]?

Dion: 2.

Teacher: How many ten-blocks are in 54 [*pointing to blocks for 54*]?

Dion: 5.

Teacher: How many ten-blocks is that altogether? [*If Dion cannot do this addition directly, ask if counting by tens can help him, because he has shown he is able to do this.*]

Dion: 7.

Teacher: How many one-blocks are in these 7 ten-blocks?

Dion: 70.

Teacher: Good. Write 70 on your paper so that you remember it.

Teacher: How many one-blocks are in 23 [*pointing to blocks for 23*]?

Dion: 3.

Teacher: How many one-blocks are in 54 [*pointing to blocks for 54*]?

Dion: 4.

Teacher: How many one-blocks is that altogether?

Dion: 7.

Teacher: How many one-blocks are there altogether [*pointing across all blocks*]?

Dion: 70 and 7. That's 77.

If Dion struggles with this type of problem, use the same procedure but with the following sequence of problem types.

1. Add multiples of ten only: 30 + 50.
2. Add a multiple of ten to a mid-decade number: 34 + 50.
3. Add two mid-decade numbers, with no regrouping: 34 + 53.
4. Add two mid-decade numbers with regrouping: 36 + 58.

For each problem, ask Dion if he can add tens first. If he cannot, have him use level 2 reasoning to count by tens. Then present another problem of the same type and again ask if he can add tens first. Do not move on to a new problem type until Dion uses level 3 reasoning on the type he is currently working on.

Note how this instructional sequence, derived from a research-based learning progression, has Dion continuously building on his current reasoning, with each developmental step that he takes small enough so that his chances of completing it are very high.

Students' Reasoning and Sense Making About the Concept of Length

To further illustrate the earlier discussion of reasoning and sense making, we examine students' reasoning about the concept of length. We look at the different ways that students are able to reason about and make sense of this topic and how instruction can encourage and support students in their development of increasingly more sophisticated levels of reasoning. There are three key steps to helping students make sense of a formal mathematical idea. First, determine empirically how they currently are making sense of the idea. Second, hypothesize how their understanding of the idea might progress. Third, choose problems and representations that can potentially help them achieve more sophisticated ways of reasoning.

The Home to School problem below (fig. 1.6) provides an excellent assessment of how well young students understand the concept of length. We first examine how students made sense of this problem, then we examine the kinds of instruction each student needs.

Which sidewalk from home to school is longer, the black one, the blue one, or are they the same (Battista 2012b)?

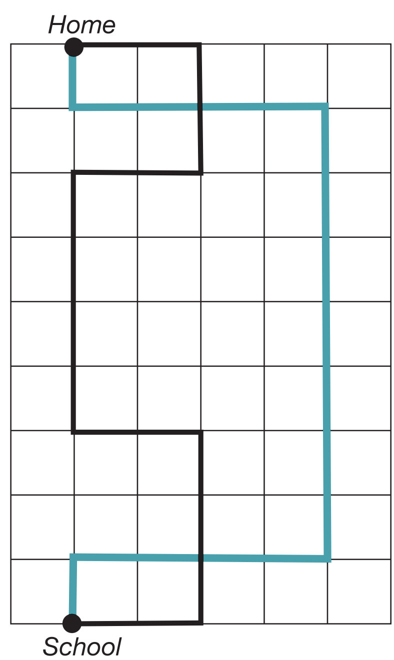

Fig. 1.6

Investigating Students' Reasoning and Sense Making about Length

Deanna says that the black sidewalk is longer because it is more "curvy." She made personal sense of the problem by relating it to her experience of walking along paths, which tends to take longer when they have many turns. (Often young students confound the length of a path with the time it takes to walk the path.)

David uses the spread between thumb and finger to draw a straight version of the blue sidewalk, one straight component at a time (right-hand drawing in fig 1.7). He uses this same procedure to construct a straight version of the black sidewalk (left-hand drawing in fig. 1.7). He then compares his drawings and says that the black sidewalk is longer. David made personal sense of the problem by straightening the paths and directly comparing them, side by side. Note that this reasoning suggests the beginnings of a valid understanding of the concept of length, and if it were done precisely (say piece by piece on a large grid with the same size units as in the problem picture), it would be mathematically correct.

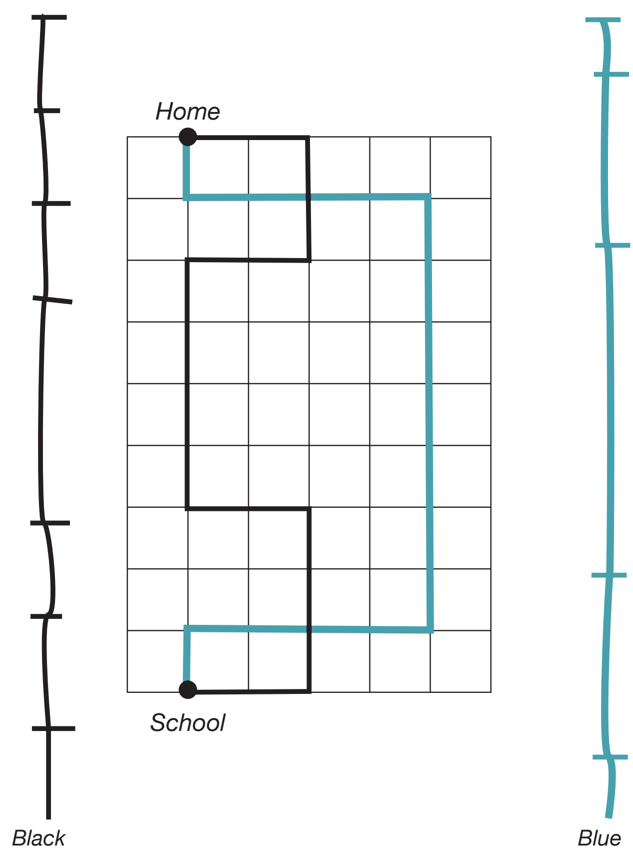

Fig. 1.7

Mollie, Matt, and Natalie make personal sense of the problem by reasoning that they should count *something*, a strategy they have often seen used in their classrooms (fig. 1.8). However, Molly and Matt do not yet understand exactly what to count. Molly counts whole (unequal) straight sections of the sidewalks and concludes that the black sidewalk is longer. Matt has observed people counting squares along paths on similar tasks, but he does not recognize how counting squares can be done in a way that corresponds to counting unit lengths; he concludes that the blue path is longer. Only Natalie correctly counts 17 unit-lengths along each sidewalk to conclude that the two paths have the same length.

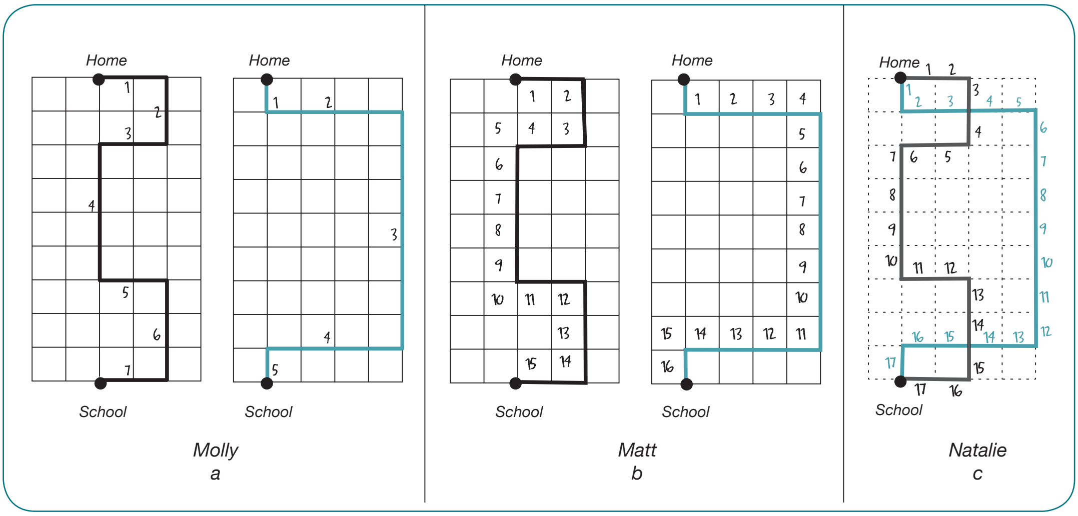

Fig. 1.8. Three students' solutions to the Home to School problem

Research Note

Difficulties in reasoning about this problem are widespread among elementary students. In individual interviews with students in grades 1–5, Battista (2010, 2012b) found that more than twice as many students used non-measurement strategies as those who used measurement strategies even though measurement strategies are most appropriate. Non-measurement strategies do not use numbers (like Deanna and David); measurement strategies use numbers (like Molly, Matt, Natalie). Furthermore, no first- or second-grade students, and only 6 percent of third graders, 12 percent of fourth graders, and 21 percent of fifth graders used correct measurement reasoning on this task (like Natalie). Even some adults have difficulty with the problem.

Instruction Focused on Individual Student Needs

Choosing instruction to help Deanna, David, Mollie, and Matt make progress in understanding length measurement requires us to understand the thinking each student can build on (see Battista 2012, for a detailed description of instructional activities). For instance, we can help Deanna by having her straighten paths and compare them directly.[4] We can help David by giving him precut sticks or rods that match the straight segment lengths on the two paths, then having him make straight paths from each set of sticks to see that the paths are the same length when carefully straightened.

We can help Molly and Matt by having them physically place unit-length rods along each path, count the unit lengths, then use the set of unit-lengths from each path to make straight versions of the paths. Because there are 17 unit-lengths in each path, when we straighten the paths, they have exactly the same length. It is important to note that most young children do not *logically* understand why counting the number of unit-lengths in the two paths tells us which is longer when straightened. They come to this conclusion *empirically* by repeatedly observing that counting predicts which path will be longer when they physically straighten the paths. It is also useful to have students compare unit-length and square iterations along grid paths to see that they generally produce different counts (unless the paths are straight). Some students make sense of this discrepancy by saying that the plastic squares have to be held "sideways"—perpendicular to the student page—to give a correct count.

The key here is to help students build on what they know to make sense of increasingly more sophisticated reasoning about length. If instruction is too abstract for students' current states of understanding, they will not be able to make the appropriate jump in personal sense making.

Integrating Conceptual and Procedural Knowledge in Reasoning About Length

Examining the conceptual and procedural knowledge needed to solve the home-to-school problem sheds additional light on student reasoning and sense making. Both types of knowledge are critical for mathematical proficiency (Kilpatrick, Swafford, and Findell 2011).

A *concept* is the meaning a person gives to objects, actions, and abstract ideas. Concepts are the building blocks of reasoning; we reason by manipulating, reflecting on, and interrelating concepts. *Conceptual knowledge* enables us to identify and define concepts, see relationships between concepts, and use concepts to reason. For instance, we might *conceptualize* length as the linear extent of an object when it is straightened or the distance you travel as you move along a path.

Procedural knowledge enables us to perform and use mathematical *procedures*, which are repeatable sequences of actions on objects, diagrams, or mathematical symbols. Procedural knowledge includes more than computational skill. For instance, to solve many length problems, students use the procedure of iterating and counting unit lengths.

The reasoning used by Molly and Matt does not properly connect conceptual and procedural knowledge. These two students have not yet developed a conceptual understanding of length measurement. Molly does not understand that the iterated units must be the same length. Matt either does not understand that the iterated units must be length units, not squares, or he thinks that iterating squares is the same as iterating unit-lengths "because they are the same size." Even more confusing for students is the fact that squares can be used to properly iterate unit-lengths for straight paths and for non-straight paths if done very carefully. Indeed, understanding the difference between iterating squares as squares and using squares to iterate unit-lengths is extremely difficult for many students. If Matt thinks that counting squares iterates unit-lengths, he may not have sufficient conceptual knowledge of unit-length iteration to regulate his use of squares to iterate unit lengths; the whole sidewalk path must be covered by a sequence of unit lengths with no gaps or overlaps in the sequence.

Summary

This discussion of the home-to-school problem illustrates that during the process of learning, students make sense of the same formal mathematical idea in different ways. For instruction to promote and support students' reasoning and sense making about length, it must be chosen to help each student build on his or her current mathematical ideas.

Observing How Instruction Can Help Students Develop the Concept of Unit-Length Iteration

We now examine how teaching can help students develop a concept of unit-length iteration that is abstract and powerful enough to deal correctly with the home-to-school problem. We first look at instruction that preceded the presentation of the home-to-school problem. Then we examine how the students made sense of this problem in light of their previously developed concepts of unit-length iteration.

Six students are working together with their teacher. The students are working on a sequence of increasingly difficult problems like the following one (fig. 1.9):

Which wire is longer, or are they the same length? Suppose I pull the wires so they are straight. Which would be longer? Can you check your answers with inch inch-rods? Can counting anything help you solve this problem? (Battista 2012b)

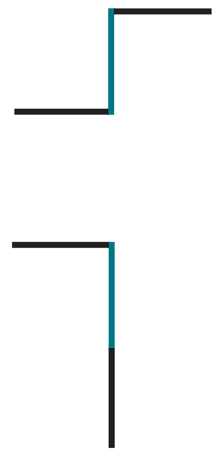

Fig. 1.9

The students made sense of and reasoned about this problem in a variety of ways.

Michael: I think they might be the same length because it's curved.

Kerri: [*Uses the eraser end of her pencil to move upward on the vertical segment of the bottom wire*] 1, 2, . . . 6, [*pointing at the horizontal segment of the bottom wire*] 7. [*Uses the eraser to count the 3 unit-segments on the top wire*] 1, 2, 3; 3 plus a little more. The bottom wire is longer.

Zack: Two on this [*positions pencil horizontally then slightly vertically on the bottom wire*]. And 3 on this [*positions the pencil horizontally, vertically, and then horizontally on the top wire*]. I think the top wire is longer. [*Note that this is the same type of straight-section reasoning that Molly used.*]

Brandi: Wait! I would say they're the same. Like this would be [*points at the bottom black vertical segment of the bottom wire*], like this [*points at the bottom black horizontal segment of the top wire (fig. 1.10)*]. And this line [*points at the blue vertical segment of the bottom wire*] would fit here [*points at the blue vertical segment of the top wire*], and this line [*points at the horizontal black segment of the bottom wire*] would fit here [*points at the top black horizontal segment of the top wire*].

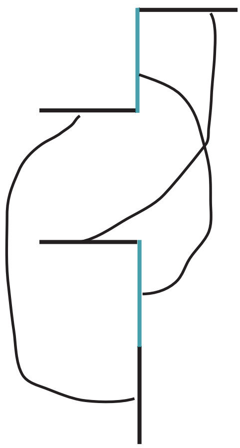

Fig. 1.10. Showing Brandi's reasoning on the wire problem

Gerald: I thought this one was longer [*top wire*], because I measured by using my fingers [*shows a finger spread*].

Kerri: The line might be an inch [*pointing at the top segment of the bottom wire*]. Then I would know that it would be 3 inches on there [*goes over the bottom wire*] and maybe 3 inches on there [*goes quickly over the top wire*]. So they're the same.

Teacher: Do you want to check with the inch-rods [*straws cut into 1-inch pieces*]? Each one of these is 1 inch long.

Kerri: 1, 2, 3 [*moves an inch-rod along the bottom wire*]. 1, 2, 3 [*moves an inch-rod along the top wire*]. They're the same length.

Teacher: Without using those inch-rods, is there anything that you could do to solve this problem by counting?

Brandi: 1, 2, 3 [*pointing to segments on the bottom wire*]. 1, 2, and 3 [*pointing to segments on the top wire*].

There is a wide variety of sophistication in students' reasoning about this problem, from Michael's non-measurement reasoning, to Kerri and Zack's incorrect counting, to Brandi's non-measurement but correct one-to-one correspondence, to Kerri's and Brandi's correct counting. To encourage and support the students in moving toward more sophisticated reasoning, the teacher not only provides students with an opportunity to check their answers, but also has them publicly discuss how they could have solved this problem by counting unit-lengths. Note, however, that some students' sense making would have benefitted from putting 3 unit-lengths on two separate wires, counting the unit-lengths, then actually straightening each wire. Also note how Brandi seemed to have made sense of Kerri's counting procedure and incorporated it into her own reasoning.

After several other problems, students are given the following problem (fig. 1.11):

Which wire is longer, or are they the same length? Suppose I pull the wires so they are straight. Which would be longer? Can you check your answers with inch-rods? Can counting anything help you solve this problem? (Battista 2012 b)

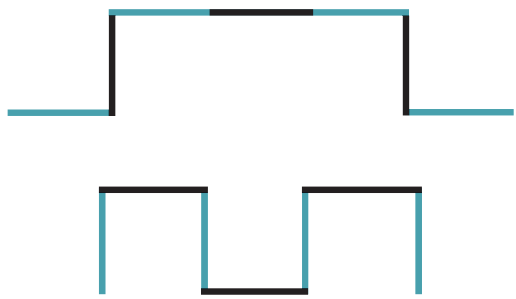

Fig. 1.11

Zack: 1, 2, 3, 4, 5, 6, 7 [*points at segments on the top wire*]; 1, 2, 3, 4, 5, 6, 7 [*points at segments on the bottom wire*]. Both are the same.

Kerri: [*Counts 7 unit-segments on each wire.*] Yep.

Serena: [*Makes a slash on each segment on both wires as she counts (fig. 1.12a).*]

Michael: [*Writes numbers on the segments for both wires as he counts (fig. 1.12b).*]

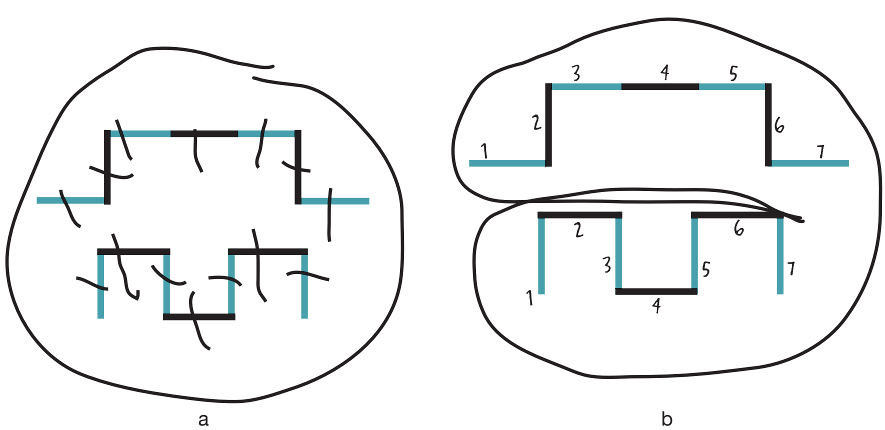

Figure 1.12

Teacher: And do you want to check them with the inch-rods?

Kerri: Yeah.

Students: [*Several students at once*] No!

By the end of this session, the students were routinely counting unit-lengths to compare the lengths of the wires in problems like those shown above.

Two weeks later the teacher returns to the topic of length and has her students work on the home-to-school problem, each with his or her own two activity sheets, one path per sheet. Although the students routinely used unit-length (inch) counting on the previous set of problems, in the new context of the home-to-school problem, students abandoned this strategy. Because the sidewalk paths are drawn on square-inch grids, squares become visually salient for the students. Their concept of unit-length iteration was not abstract and general enough to apply in this new situation. The dialogue below, which took place after the students had given the strategies described above, illustrates how the students' sense making evolved.

Teacher: When we're trying to figure out the lengths of the sidewalks, what should we count?

Gwen: I think we should count the squares because they're like an inch.

Kerri: The squares are as long as the segments [*points at a square along a sidewalk, then at its side*]. So they're the same length, which means that, if you chose either one of them it wouldn't be wrong because they're the same length. [*Pause*] Well, you might not come up with the same answer. 'Cause there's more squares than segments. Oh wait! Then you could just like count the squares that are nearest [*pointing at the sidewalk*].

Gwen: And you wouldn't count ones near the corner because they're not near a segment; it's just a corner touching the line.

Kerri: You would like want to count all the ones that have a segment on them [*points at a segment on a sidewalk path*].

Teacher: [*Deciding that the students should all be looking at the same thing, shows Serena's sheet for the black path*] Now you're saying that this part of the sidewalk is 4 blocks [*points to the squares numbered 1–4 in fig. 1.13*], right?

Teacher: What would happen if we were measuring this and we used our 1-inch straws?

Gwen: That's a problem. You can't count the squares because like they would be sharing one; this and this [*pointing at the second and third unit-segments, starting at Home; each needs a square and we*

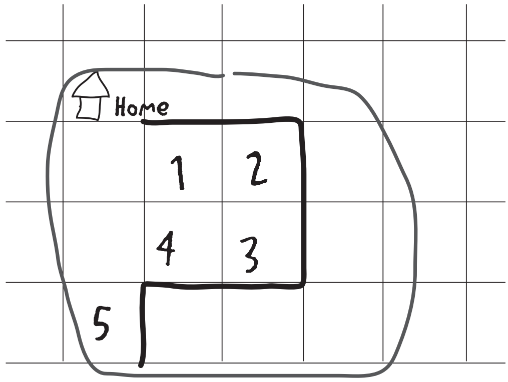

Fig. 1.13

	only counted one (the 2)]. So you would need to count the sides. So we would count 1, 2, 3, 4, 5, 6, and 7 around here [*correctly counting unit-lengths on the section of the sidewalk shown in fig. 1.13*].
Teacher:	So how long do you think the whole black sidewalk is?
Gwen:	[*Correctly pointing to and counting unit lengths along the black sidewalk*] 1, 2, . . . , 16, 17.
Teacher:	Seventeen what?
Gwen:	Sides, same as our inch-straws.

Note the different ways that students made sense of this problem during the discussion. At first, Kerri thought that using squares in the grid would work because "the squares are as long as the segments." Kerri revised her reasoning and then claimed that they should count only the squares that have a side on the sidewalk path. When Serena and Gwen seemed to do what Kerri suggested (fig. 1.9), they came up with an incorrect count. So the teacher asked a question that she thought might provoke students to revise their reasoning. Gwen then realized the difficulty with counting squares; she made sense of the correct method for iterating unit-lengths along the sidewalk paths.

This episode is an excellent example of SMP 2, reason abstractly and quantitatively. The students had to decontextualize the unit-length counting strategy they used in the rod problems and re-contextualize it (transfer it) to apply it in the more difficult and complex context of the home-to-school problem. This re-contextualizing did not occur automatically; it required reasoning and sense making above and beyond the reasoning they had applied in previous problems. In fact, it was reasoning about this new problem that led students to construct a more powerful concept of unit-length iteration that could be applied to more complex situations. Most often, students' initial reasoning is context dependent; it is only by giving students a variety of contexts that students decontextualize and abstract the reasoning so that it becomes generally applicable.

Standards for Mathematical Practice and Process Standards in a Sense-Making Episode

To relate our discussion of students' reasoning about and sense making of the concept of length to the CCSSM Standards for Mathematical Practice and NCTM's Process Standards, we explicitly examine how the episodes on length are related to these practices and processes.

Standards for Mathematical Practice

Students clearly tried to make sense of the problems. Not only did the sense making differ among students but it also evolved over instructional time as students made sense of the concept of length by straightening paths, matching equal sub-lengths, and finally by ever increasingly more sophisticated counting (SMP 1a, b, g). They translated between different representations—numerical counting and spatial-unit iteration—both concretely and pictorially (SMP 1f). They made ever-increasing sense of counted quantities (SMP 2a). They constructed and evaluated arguments, and gave explanations and justifications for their work (SMP 3a, d, f). They applied the mathematics of counting and the concept of length to a real-world situation depicted in the home-to-school problem (SMP 4a). They identified important quantities and made sense of the numerical results as their notions of what must be counted evolved (SMP 4c, d). They used appropriate inch-rod tools (SMP 5). They attended to precision as they moved away from eraser estimations and toward methods of counting that were relevant to the problem, in essence creating, in action, a definition for the appropriate unit to enumerate (SMP 6b). They communicated precisely (SMP 6a). They saw structure when they used one-to-one matching of unit-segments in two wires (SMP 7)—they saw that both wires were made from the same set of linear components and thus had the same linear structure, and thus the same length. Finally, they continually evaluated their methods (SMP 8d).

Process Standards

Clearly the students built new mathematical knowledge through problem solving by implementing, discussing, and evaluating solution strategies (PS 1a, b). They reasoned and justified, developed mathematics arguments, and communicated and evaluated their thinking and strategies (PS 2a, 2c, 3a, 3b). They connected counting and spatial iteration and applied mathematics (PS 4a, b). They represented spatial unit-length iteration with counting (PS 5a, b). As shown in figure 1.14, some students even progressed from counting to reasoning using addition and fractions (PS 4a, 5b).

Kerri: I separated this one in half [*draws a vertical segment separating the bottom wire into two parts (fig. 1.14)*], and I knew 4 + 4 was 8. And the top wire is 4 + 4 [*circling the right and left sides of the top wire*] plus 1 in the middle [*pointing*]. The top is longer.

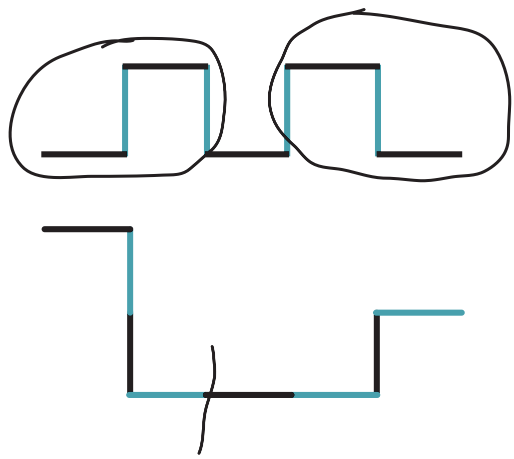

Fig. 1.14

Concluding Remarks on Mathematical Reasoning and Sense Making

To use mathematics to make sense of the world, students must first make sense of mathematics. To make sense of mathematics, students must transition from intuitive, informal reasoning stemming from their interactions with the world to precise reasoning that uses formal mathematical concepts, procedures, and symbols. The key to helping students make this transition is providing appropriate instructional tasks that target precisely those concepts and ways of reasoning that students are ready acquire. And the key to providing this support is an understanding of research-based descriptions of the development of students' increasingly more sophisticated conceptualizations and reasoning about particular mathematical concepts. Understanding the development of students' mathematical thinking is critical for selecting and creating instructional tasks, asking appropriate questions of students, guiding classroom discussions, adapting instruction to students' needs, understanding students' reasoning, assessing students' learning progress, and diagnosing and remediating students' learning difficulties.

Endnotes

1. Much of the research and development referenced in this chapter was supported in part by the National Science Foundation under Grant Numbers 0099047, 0352898, 554470, 838137, and 1119034. The opinions, findings, conclusions, and recommendations, however, are mine and do not necessarily reflect the views of the National Science Foundation.
2. An additional learning progression for the development of another aspect of number understanding is described in chapter 2, and several learning progressions for geometry are described in chapter 4.
3. It is assumed that all students pass through almost all of the levels in learning progressions. What varies is the speed at which they pass through the levels and the amount of instructional scaffolding students need to pass through each level.
4. For instruction on this task, draw the paths on square-inch grid paper and have available individual inch-rods. Also useful are sets of inch-rods strung on flexible wires so that students can make the problem paths and straighten them.

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