



GRADE 4

# PROBLEM SOLVING *and* REASONING

## Introduction

*Principles and Standards for School Mathematics* (NCTM 2000) states that “problem solving is the cornerstone of school mathematics. Without the ability to solve problems, the usefulness and power of mathematical ideas, knowledge, and skills are severely limited” (p. 182). Mathematical investigations that challenge students to deal with nonroutine problems and situations should be a regular part of Standards-based instruction at all levels.

Just as solving problems can help students make sense of their changing world, justifying solutions and communicating the results of mathematical investigations can help elementary school students develop and expand their reasoning abilities. One goal is for students to develop ways of thinking about mathematics that encourage sense making and reasoning about solutions and strategies. The mathematics classroom is the main environment in which students speak and write mathematics. Hence, it is essential that teachers offer students opportunities to communicate mathematically by having them make, test, discuss, and refine conjectures, ultimately accepting or rejecting them.

The investigations in this book engage students in extended tasks that enable them to look for relationships among concepts in the five content strands—number and operations, algebra, geometry, measurement, and data analysis and probability. Each investigation enables students to focus on one strand in depth. At the same time, the investigations illustrate how a carefully chosen mathematical task can bridge content areas. For example, although the focus of the data analysis investigation is on analyzing data and drawing conclusions from them, students work with numerical computations involving averaging, working with decimal numbers, and making metric measurements.

## Aspects of Problem Solving

Good problems challenge students to develop and apply strategies, serve as a means to introduce new concepts, and offer a context for using skills. Problem solving is not a specific topic to be taught but a process that permeates all mathematics.

What behaviors might a teacher expect to observe in a classroom that makes problem solving a focus? According to *Principles and Standards for School Mathematics*, all students should—

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems; and
- monitor and reflect on the process of mathematical problem solving. (NCTM 2000, p. 402)

Fourth-grade students have an opportunity to engage in problem solving as they complete the investigations in this book, discuss their ideas and conjectures in pairs or small groups, and justify their thinking to the teacher and other members of the class. As teachers facilitate the investigations, the tasks naturally lead to such questions as “Why?” and “How do you know?”

### Students build new mathematical knowledge through problem solving

Students can learn new mathematical concepts and skills through problem solving. A successful problem-centered approach uses interesting problems to motivate students to spend time and energy and be persistent in seeking solutions. Under the guidance of a teacher who encourages students to reason creatively and make connections between ideas, students can discover new mathematical concepts, techniques, and relationships. New ideas often emerge from discussions among students. Teachers should guide such discussions carefully so that the students learn the difference between correct mathematical reasoning and incorrect reasoning and between sound problem-solving strategies and unsound ones. The teacher must summarize classroom discussions so that the students are aware of the new knowledge and skills that they have derived from the problem-solving experience.

### Students solve problems that arise in mathematics and other contexts

The investigations in this book pose problems to solve in contexts that are mathematically rich, appeal to fourth graders, and facilitate communication skills. In *Discovering Primes As the Ancient Mathematicians Did*, students begin with a scenario in which some other students are trying to solve a challenging problem to get their teacher to participate in a dunking booth at a carnival. In *Movie Money Matters*, students explore contracts between theater owners and movie distributors and, in the process, calculate mathematical change over



*“Students who can both develop and carry out a plan to solve a mathematical problem are exhibiting knowledge that is much deeper and more useful than simply carrying out a computation.”*  
(NCTM 2000, p. 182)



time. Making and Investigating Puzzles gives students experience in decomposing and recomposing two-dimensional puzzles. In Fascinating Measures, students find their heart rate for a minute and use the results to determine their number of heartbeats in a year and predict their number of heartbeats in a lifetime. In Growing Giant Sequoias, students interpret a set of data to determine the lighting conditions that appear to improve the growth of tree seedlings. In each of these investigations, students grapple with problems that arise in real-world situations. The problems are interesting and challenging vehicles for exploring mathematics and thinking about relationships.

## Students apply and adapt a variety of strategies to solve problems

As students explore problems, they need to consider a variety of strategies to investigate the solution. In Movie Money Matters, students represent profits by plotting points and creating bar graphs and tables. In Growing Giant Sequoias, different groups of students use different approaches to extrapolate data for missing tree heights and represent their approaches through graphs or tables. The varied representations afford students many different ways to explore a problem and enable those with different learning styles to benefit from the problem-solving experience.

## Students monitor and reflect on the process of mathematical problem solving

As students work through good mathematical tasks, they reflect on their work to determine what strategies are effective and where they need to make adjustments. In Making and Investigating Puzzles, students investigate cutting two-dimensional figures to produce specific new shapes, and they investigate puzzle pieces to determine which ones have specific characteristics (e.g., line symmetry or rotation symmetry). Students are encouraged to make a conjecture and then test it. By monitoring what happens when they try to fit a piece in the puzzle, students become much more proficient in making conjectures. In Fascinating Measures, students use a multistep problem-solving process to determine the number of heartbeats in an hour, a day, or a year. They then apply this same process to other measurement facts to represent the measure in an equivalent way that is easier to understand. By reflecting on the process that they use with one problem, students can build a repertoire of problem-solving strategies that they can apply to a wide range of problems.

## Aspects of Reasoning

Reasoning develops over time as teachers facilitate discussion of rich tasks and help students learn “to construct valid arguments and to evaluate the arguments of others” (NCTM 2000, p. 188). As students reason about mathematics, they should—



*“Reflecting on different ways of thinking about and representing a problem solution allows comparisons of strategies and consideration of different representations.”  
(NCTM 2000, p. 185)*



*“Good problems and problem-solving tasks encourage reflection and communication and can emerge from the students’ environment or from purely mathematical contexts.”  
(NCTM 2000, p. 183)*



*“During these grades [3–5], formulating conjectures and assessing them on the basis of evidence should become the norm.”*  
(NCTM 2000, p. 188)



*“Mathematical reasoning develops in classrooms where students are encouraged to put forth their own ideas for examination. Teachers and students should be open to questions, reactions, and elaborations from others in the classroom.”* (NCTM 2000, p. 188)



*“At these grades [3–5], students need experiences in learning about what constitutes a convincing argument.”* (NCTM 2000, p. 190)



- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs; and
- select and use various types of reasoning and methods of proof. (NCTM 2000, p. 402)

## Students make and investigate mathematical conjectures

Students need to learn that making conjectures on the basis of patterns is a natural part of mathematical thinking. In *Discovering Primes As the Ancient Mathematicians Did*, students use the Sieve of Eratosthenes to find how many prime numbers exist between 1 and 100. In *Making and Investigating Puzzles*, students think about the properties of puzzle pieces and make a conjecture about whether a piece will fit before actually trying it.

## Students develop and evaluate mathematical arguments and proofs

In *Movie Money Matters*, students compare different contracts between theater owners and movie distributors and discuss which contracts should be renegotiated. In *Growing Giant Sequoias*, students consider growing tree seedlings in sun or shade, develop a process to determine which lighting condition is better, present their process to the class, and then adjust it on the basis of feedback from other students.

## Students select and use various types of reasoning and methods of justification

Throughout the investigations, students have opportunities to justify their reasoning in several ways. The algebra investigation, *Movie Money Matters*, prompts students to evaluate different contracts to determine which one provides the largest profits for the theater owners and the movie distributors. In the geometry investigation, *Making and Investigating Puzzles*, students reason by making conjectures and then testing those conjectures on the computer or by physically making puzzle pieces. The data analysis investigation, *Growing Giant Sequoias*, encourages students to reason by applying a proposed plan to evaluate which lighting condition is better for the growth of tree seedlings.

## The Role of the Teacher

As students explore the tasks in this book, their teachers should monitor their activities and foster the interactions necessary to maintain high levels of reasoning. (See Stein and Smith [1998]). The tasks are mathematically rich, but if teachers provide too many clues or too much specific help early in the process, they can stifle the deep thinking

that the tasks can elicit from students. The challenge for teachers is to facilitate students' communication about a task without directing the students toward a particular solution.

From detailed observations in an elementary school classroom using a Standards-based curriculum, Fraivillig (2001) has identified various strategies that are essential to helping students think deeply about mathematical ideas and share their thinking with others. These approaches fall into three broad categories: eliciting students' thinking, supporting students' thinking, and extending students' thinking. Descriptions of the strategies in each category are summarized in figure 1.

### Strategies to elicit students' thinking

- Elicit many solution methods for one problem.
- Wait for, and listen to, students' descriptions of solution methods.
- Encourage students to elaborate and discuss.
- Use students' explanations as a basis for the lesson's content.
- Convey an attitude of acceptance of students' errors and efforts.
- Promote collaborative problem solving.

### Strategies to support students' thinking

- Remind students of conceptually similar problems.
- Provide background knowledge.
- Lead students through "instant replays." (Revisit student solutions.)
- Write symbolic representations of solutions when appropriate.

### Strategies to extend students' thinking

- Maintain high standards and expectations for all students.
- Encourage students to make generalizations.
- List all solution methods on the board to promote reflection.
- Push individual students to try alternative solution methods.
- Promote the use of more efficient solution methods.

*Adapted from Fraivillig (2001, pp. 454–59)*

Fig. 1.

Strategies to advance students' thinking



See Fraivillig (2001) on the CD-ROM for classroom

strategies that elicit, support, and extend students' thinking.



See Henningsen and Stein (1997), Smith and Stein (1998), and Stein

and Smith (1998) on the CD-ROM for ideas on using mathematical tasks to stimulate reflection.

In addition, research from the Quantitative Understanding Amplifying Student Achievement and Reasoning (QUASAR) Project in urban schools with underachieving students found that the following actions by teachers were associated with higher performance by their students on a test of problem solving (Henningsen and Stein 1997; Stein, Grover, and Henningsen 1996; Smith and Stein 1998):

- Teachers press for explanations and meaning.
- Teachers have capable students model high-level performance.
- Teachers allow appropriate time for students to explore the task, think, and make sense of mathematics for themselves.
- Teachers note conceptual connections.
- Teachers build on students' prior knowledge.

Teachers who engage in behaviors like those identified by Fraivillig and the QUASAR researchers can help students develop their reasoning and problem-solving abilities. Teachers can use the following questions to help elicit students' reasoning:

- “Why?”
- “How do you know?”
- “What other problems can you remember that are similar to this one?”
- “Are there other ways you could solve this problem?”
- “Do you agree with this approach to this problem? Why, or why not?”

Such questions can stimulate important teacher-student discourse that will strengthen the reasoning abilities of all students and can engage students in mathematical communication in the classroom.

The role of the teacher is indispensable, and the investigations in this book are designed to help teachers encourage problem solving and reasoning by elementary school mathematics students. Engaging students in these processes is an essential component of developing their mathematical power.